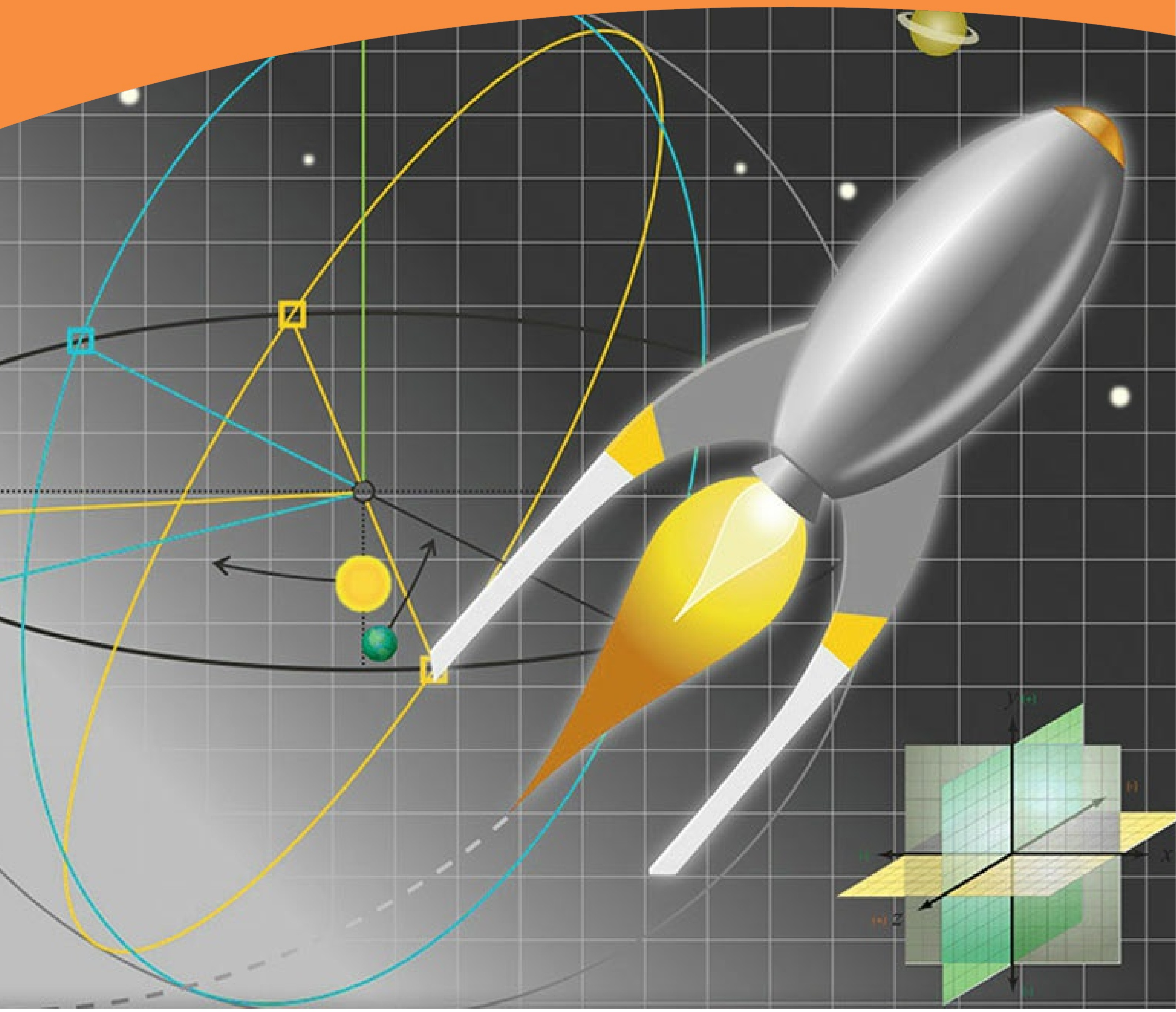
Modern Introductory Mechanics Part II

Walter Wilcox



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MODERN INTRODUCTORY MECHANICS PART II

Modern Introductory Mechanics Part II
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8 PARTICLE INTERACTIONS AND CENTRAL FORCES

8.1 MULTI-PARTICLE CONSERVATION LAWS

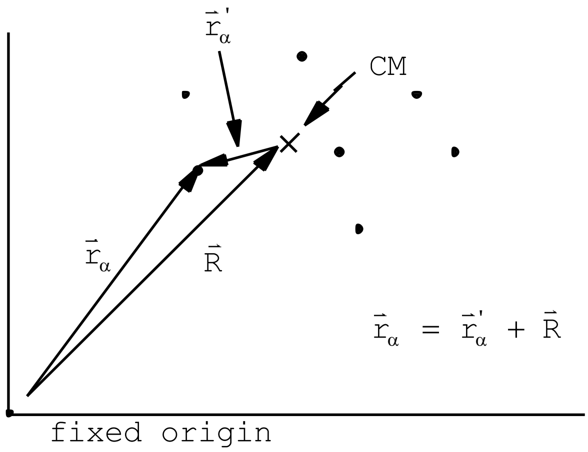
Multiparticle conservation theorems now. A generalization of results for momentum and energy reached in Ch.2 for a single particle.

Latin indices (a,b,c, etc.): vector indices
Greek indices ($\alpha,\beta,\gamma, \ldots$): particle number

Reminder of conserved quantities:

Invariance of L	conserved quantity
$\vec{r}_\alpha \to \vec{r}_\alpha + \delta\vec{r}$	$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha$
$\vec{r}_\alpha \to \vec{r}_\alpha + \left(\delta\vec{\theta} \times \vec{r}_\alpha\right)$	$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha$
$t \to t + \delta t$	$H \; (= T + U)$

Consider an arbitrary system of particles:



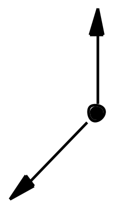
Center of mass located by $\left(M = \sum_\alpha m_\alpha\right)$

$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha \,. \tag{8.1}$$

We have

$$\sum_\alpha m_\alpha \vec{r}_\alpha^{\,'} = \sum_\alpha m_\alpha\left(\vec{r}_\alpha - \vec{R}\right) = M\vec{R} - M\vec{R} = 0\,. \tag{8.2}$$

Particle α:



$\vec{f}_{\alpha\beta}$: internal force on α due to β

$\vec{F}_\alpha^{(e)}$: external force

Total force on α:

$$\vec{F}_\alpha = \vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta} \,. \tag{8.3}$$

Call $\vec{f}_\alpha = \sum_\beta \vec{f}_{\alpha\beta}$. From Newton's third law

$$\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}. \text{ (weak form)} \tag{8.4}$$

Also assume

$$\vec{f}_{\alpha\alpha} = 0 \,. \tag{8.5}$$

(Newton's mechanics not equipped to handle self-interactions which, however, really do exist!) Newton's second law:

$$\dot{\vec{p}}_\alpha = \vec{F}_\alpha^{(e)} + f_\alpha \,, \tag{8.6}$$

or

$$\frac{d^2}{dt^2}\left(m_\alpha \vec{r}_\alpha\right) = \vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta} \,. \tag{8.7}$$

Sum on α:

$$\frac{d^2}{dt^2}\left(\sum_\alpha m_\alpha \vec{r}_\alpha\right) = \sum_\alpha \vec{F}_\alpha^{(e)} + \sum_{\alpha,\beta} \vec{f}_{\alpha\beta} \,,$$

$$\sum_{\alpha,\beta} \vec{f}_{\alpha\beta} = \underbrace{\vec{f}_{12} + \vec{f}_{21}}_{0} + \ldots = 0 \,,$$

$$\Rightarrow \frac{d^2}{dt^2}\left(M\vec{R}\right) = \sum_\alpha \vec{F}_\alpha^{(e)} \equiv \vec{F}^{(e)} \,. \tag{8.8}$$

Center of mass moves as if the total external force were acting on the entire mass of system concentrated at the center of mass. (Internal forces have no effect on CM motion.) Of course, if $\vec{F}^{(e)} = 0$, then

$$\frac{d^2}{dt^2}\left(M\vec{R}\right) \equiv \dot{\vec{P}} = 0\,, \tag{8.9}$$

and total linear momentum is conserved.

Likewise, the angular momentum of the $\alpha^{\underline{th}}$ particle about the origin is

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha\,. \tag{8.10}$$

Summing this:

$$\vec{L} = \sum_\alpha \vec{L}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha \left(\vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha\right)$$

$$= \sum_\alpha \left(\vec{r}_\alpha^{\,'} + \vec{R}\right) \times m_\alpha \left(\dot{\vec{r}}_\alpha^{\,'} + \dot{\vec{R}}\right). \tag{8.11}$$

$$\vec{L} = \vec{R} \times \vec{P} + \sum_\alpha \vec{r}_\alpha^{\,'} \times \vec{p}_\alpha^{\,'}. \qquad \left(\vec{p}_\alpha^{\,'} \equiv m_\alpha \dot{\vec{r}}_\alpha^{\,'}\right) \tag{8.12}$$

Total angular momentum about a coordinate axis is the angular momentum of the system as if it were concentrated at the center of mass, plus the angular momentum of motion about the center of mass. Now the rate of change of $\vec{L}_\alpha$ is

$$\dot{\vec{L}}_\alpha = \vec{r}_\alpha \times \dot{\vec{p}}_\alpha = \vec{r}_\alpha \times \left(\vec{F}_\alpha^{(e)} + \sum_\beta \vec{f}_{\alpha\beta}\right)$$

$$\Rightarrow \dot{\vec{L}} = \sum_\alpha \dot{\vec{L}}_\alpha = \underbrace{\sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{(e)}}_{①} + \underbrace{\sum_{\alpha,\beta} \vec{r}_\alpha \times \vec{f}_{\alpha\beta}}_{②}\,. \tag{8.13}$$

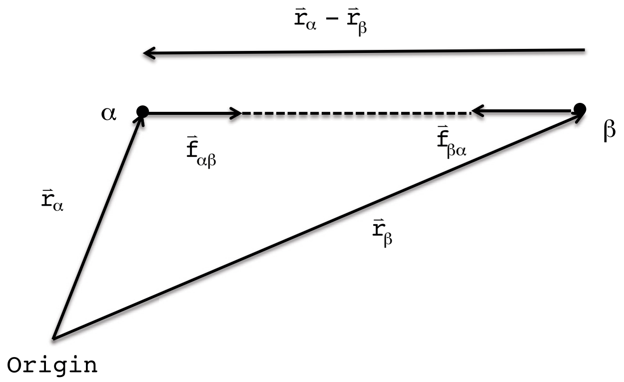
(assume $\vec{f}_{\alpha\alpha} = 0$)

$$① = \sum_\alpha \underbrace{\vec{r}_\alpha \times \vec{F}_\alpha^{(e)}}_{\vec{N}_\alpha^{(e)}} \equiv \vec{N}^{(e)}, \tag{8.14}$$

$$② = -\sum_{\alpha,\beta} \vec{r}_\alpha \times \vec{f}_{\beta\alpha} = -\sum_{\alpha,\beta} \vec{r}_\beta \times \vec{f}_{\alpha\beta}\,. \tag{8.15}$$

Therefore, we may write

$$② = \frac{1}{2}\sum_{\alpha,\beta}\left(\vec{r}_\alpha - \vec{r}_\beta\right) \times \vec{f}_{\alpha\beta} = 0\,. \quad \text{(see figure below)} \tag{8.16}$$



I am showing an attractive force case. I am also assuming the "strong form" of Newton's second law, which says that $\vec{f}_{\alpha\beta}$ and $\vec{f}_{\beta\alpha}$ lie along the line connecting the two particles. (Violated, for example, in electromagnetism.) Under these circumstances then

$$\dot{\vec{L}} = \vec{N}^{(e)}. \tag{8.17}$$

So $\vec{L}$ = const. in time if $\vec{N}^{(e)} = 0$.

For the kinetic energy, we have

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2}\sum_\alpha m_\alpha \left(\dot{\vec{R}} + \dot{\vec{r}}_\alpha^{\,\prime}\right)^2$$

$$= \frac{1}{2} M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,\prime 2}. \tag{8.18}$$

Kinetic energy, like angular momentum, consists of two parts: the kinetic energy of the center of mass, plus the kinetic energy of motion about the center of mass.

Now, what about work being done on the system? Define

$$W_{12} \equiv \sum_\alpha \int_1^2 \vec{F}_\alpha \cdot d\vec{r}_\alpha = \sum_\alpha \int_1^2 m_\alpha \ddot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha \, dt$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \int_1^2 \frac{d}{dt}\dot{\vec{r}}_\alpha^{\,2} dt = \sum_\alpha \frac{1}{2} m_\alpha \left(\vec{v}_{2\alpha}^{\,2} - \vec{v}_{1\alpha}^{\,2}\right),$$

$$\Rightarrow \quad W_{12} = T_2 - T_1 \quad \text{(useful in a bit)} \tag{8.19}$$

Assume conservative external and internal forces:

$$\vec{F}_\alpha^{(e)} = -\vec{\nabla}_\alpha U_\alpha(\vec{x}_\alpha), \tag{8.20}$$

$$\vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}\left(\left|\vec{r}_\alpha - \vec{r}_\beta\right|\right) \tag{8.21}$$

Then, alternatively

$$W_{12} = \sum_\alpha \int_1^2 \vec{F}_\alpha \cdot d\vec{r}_\alpha = \underbrace{\sum_\alpha \int_1^2 \vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha}_{①} + \underbrace{\sum_{\alpha,\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha}_{②}. \tag{8.22}$$

$$① = -\sum_\alpha \int_1^2 \left(\vec{\nabla}_\alpha U_\alpha\right) \cdot d\vec{r}_\alpha = -\sum_{\alpha,i} \int_1^2 \frac{\partial U_\alpha}{\partial x_{\alpha,i}} dr_{\alpha,i}$$

$$= -\sum_\alpha \int_1^2 dU_\alpha = \sum_\alpha \left(U_{1\alpha} - U_{2\alpha}\right). \tag{8.23}$$

② is more complicated. On one hand

$$② = \sum_{\alpha,\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha = -\sum_{\alpha,\beta} \int_1^2 \vec{f}_{\beta\alpha} \cdot d\vec{r}_\alpha = -\sum_{\alpha,\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\beta,$$

$$\Rightarrow ② = \frac{1}{2} \sum_{\alpha,\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot \left(d\vec{r}_\alpha - d\vec{r}_\beta\right). \tag{8.24}$$

On the other hand the chain rule gives

$$d\bar{U}_{\alpha\beta} = \sum_i \left[\underbrace{\frac{\partial \bar{U}_{\alpha\beta}}{\partial r_{\alpha,i}} dr_{\alpha,i}}_{\left(\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}\right) \cdot d\vec{r}_\alpha} + \underbrace{\frac{\partial \bar{U}_{\alpha\beta}}{\partial r_{\beta,i}} dr_{\beta,i}}_{\left(\vec{\nabla}_\beta \bar{U}_{\alpha\beta}\right) \cdot d\vec{r}_\beta}\right],$$

$$\Rightarrow d\bar{U}_{\alpha\beta} = \underbrace{\left(\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}\right)}_{-\vec{f}_{\alpha\beta}} \cdot d\vec{r}_\alpha + \underbrace{\left(\vec{\nabla}_\beta \bar{U}_{\alpha\beta}\right)}_{-\vec{f}_{\beta\alpha} = \vec{f}_{\alpha\beta}} \cdot d\vec{r}_\beta,$$

$$\Rightarrow d\bar{U}_{\alpha\beta} = -\vec{f}_{\alpha\beta} \cdot \left(d\vec{r}_\alpha - d\vec{r}_\beta\right). \tag{8.25}$$

$$\Rightarrow ② = \frac{1}{2} \sum_{\alpha,\beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot \left(d\vec{r}_\alpha - d\vec{r}_\beta\right) = -\frac{1}{2} \sum_{\alpha,\beta} \int_1^2 d\bar{U}_{\alpha\beta},$$

$$\Rightarrow ② = -\frac{1}{2} \sum_{\alpha,\beta} \left(\bar{U}_{2\alpha\beta} - \bar{U}_{1\alpha\beta}\right). \tag{8.26}$$

Therefore

$$W_{12} = \left(-\sum_{\alpha} U_{\alpha} - \frac{1}{2} \sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \bar{U}_{\alpha\beta} \right) \Bigg|_1^2 . \tag{8.27}$$

Define the total potential energy,

$$U \equiv \sum_{\alpha} U_{\alpha} + \sum_{\alpha<\beta} \bar{U}_{\alpha\beta} = \sum_{\alpha} U_{\alpha} + \frac{1}{2} \sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \bar{U}_{\alpha\beta} , \tag{8.28}$$

then

$$W_{12} = -U \Big|_1^2 = U_1 - U_2 . \tag{8.29}$$

Combining this with $W_{12} = T_2 - T_1$,

$$\Rightarrow \quad T_1 + U_1 = T_2 + U_2 . \tag{8.30}$$

Total energy, kinetic + potential, is conserved. The n-particle Lagrange function in CM coordinates can then be written:

$$L = T - U = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'^{\,2} - \sum_\alpha U_\alpha - \sum_{\alpha<\beta} \overline{U}_{\alpha\beta}\left(\left|\vec{r}_\alpha' - \vec{r}_\beta'\right|\right). \tag{8.31}$$

8.2 TWO-BODY RELATIVE COORDINATES

Let us now specialize to the 2-body problem with no external forces:

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}m_1\dot{\vec{r}}_1'^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2'^{\,2} - \underset{\substack{\uparrow \\ \text{drop the bar, subscripts}}}{U}\left(\left|\vec{r}_1' - \vec{r}_2'\right|\right) \tag{8.32}$$

Of course $\vec{r}_1'$ and $\vec{r}_2'$ are not independent, but satisfy

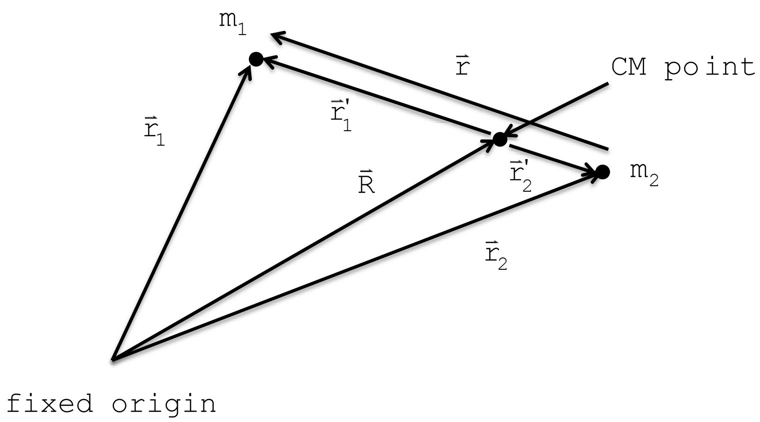
$$\sum_\alpha m_\alpha \vec{r}_\alpha' = 0 \Rightarrow m_1\vec{r}_1' + m_2\vec{r}_2' = 0. \tag{8.33}$$

(There are always 3 + 3 independent coordinates for a 2-body system) Let us introduce

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 = \vec{r}_1' - \vec{r}_2' \tag{8.34}$$

(no subscripts
or primes on $\vec{r}$)

Picture:



Then

$$\vec{r} = \vec{r}_1'\left(1 + \frac{m_1}{m_2}\right) \Rightarrow \dot{\vec{r}}_1' = \frac{m_2}{m_1 + m_2}\dot{\vec{r}}, \tag{8.35}$$

or

$$\vec{r} = -\vec{r}_2^{\,'}\left(1 + \frac{m_2}{m_1}\right) \Rightarrow \dot{\vec{r}}_2^{\,'} = -\left(\frac{m_1}{m_1 + m_2}\right)\dot{\vec{r}} . \tag{8.36}$$

Therefore

$$\frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \frac{1}{2}\left[m_1\left(\frac{m_2}{m_1 + m_2}\right)^2 + m_2\left(\frac{m_1}{m_1 + m_2}\right)^2\right]\dot{\vec{r}}^{\,2} . \tag{8.37}$$

so

$$\frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} , \tag{8.38}$$

where

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} , \qquad \text{(“reduced mass”)}$$

$$\Rightarrow L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) . \tag{8.39}$$

Now find the Hamiltonian for the above L. Adopt R_i, r_i as generalized coordinates.

$$\frac{\partial L}{\partial \dot{R}_i} = M\dot{R}_i = P_i , \qquad \left(\vec{P} = M\dot{\vec{R}}\right) \tag{8.40}$$

$$\frac{\partial L}{\partial \dot{r}_i} = \mu\dot{r}_i \equiv p_i , \qquad \left(\vec{p} = \mu\dot{\vec{r}}\right) \tag{8.41}$$

↑
small $\vec{p}$; no indices

$$H = \sum_i \frac{\partial L}{\partial \dot{R}_i}\dot{R}_i + \sum_i \frac{\partial L}{\partial \dot{r}_i}\dot{r}_i - L , \tag{8.42}$$

$$H = \underbrace{\frac{\vec{P}^2}{2M}}_{H_{cm}} + \underbrace{\frac{\vec{p}^2}{2\mu} + U(r)}_{H_{rel}} . \tag{8.43}$$

H is a constant of the motion since it has no t dependence.

$$H_{CM} = \frac{\vec{P}^2}{2M} , \qquad H_{rel} = \frac{\vec{p}^2}{2\mu} + U(r) .$$

↑ const. of motion ↑ also const. of motion

The total angular momentum is

$$\vec{L} = \underbrace{\vec{R} \times \vec{P}}_{\vec{L}_{CM}} + \underbrace{\vec{r}_1^{\,'} \times \vec{p}_1^{\,'} + \vec{r}_2^{\,'} \times \vec{p}_2^{\,'}}_{\vec{\ell}} \,. \tag{8.44}$$

(relative angular momentum)

Can rewrite $\vec{\ell}$:

$$\vec{p}_1^{\,'} = m_1\dot{\vec{r}}_1^{\,'} = m_1\left(\frac{m_2}{m_1 + m_2}\right)\dot{\vec{r}} = \mu\dot{\vec{r}} \tag{8.45}$$

$$\vec{p}_2^{\,'} = m_2\dot{\vec{r}}_2^{\,'} = m_2\left(-\frac{m_1}{m_1 + m_2}\right)\dot{\vec{r}} = -\mu\dot{\vec{r}}\,, \tag{8.46}$$

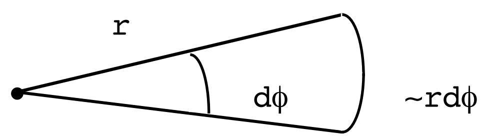
$$\begin{aligned} \Rightarrow \vec{\ell} = \vec{r}_1^{\,'} \times \vec{p}_1^{\,'} + \vec{r}_2^{\,'} \times \vec{p}_2^{\,'} &= \frac{m_2}{m_1 + m_2}\,\vec{r} \times \left(\mu\dot{\vec{r}}\right) \\ &+ \frac{m_1}{m_1 + m_2}\,\vec{r} \times \left(\mu\dot{\vec{r}}\right), \end{aligned} \tag{8.47}$$

$$\vec{\ell} = \frac{m_1 + m_2}{m_1 + m_2}\,\vec{r} \times \left(\mu\dot{\vec{r}}\right) = \vec{r} \times \vec{p}\,, \tag{8.48}$$

$$\Rightarrow \vec{L} = \vec{R} \times \vec{P} + \vec{r} \times \vec{p}. \tag{8.49}$$

Already know $\dot{\vec{L}} = 0$. But $\dot{\vec{L}}_{CM} = \dot{\vec{R}} \times \vec{P} = 0$ also $\Rightarrow \dot{\vec{\ell}} = 0$. This has 2 important consequences:

1. Since $\vec{\ell}$ is time independent and because $\vec{\ell} \cdot \vec{p} = \vec{\ell} \cdot \vec{r} = 0$ ($\perp$ to both $\vec{p}$ and $\vec{r}$), it is clear that the motion is in a plane $\perp$ to $\vec{\ell}$.
2. $\ell \equiv |\,\vec{r} \times \vec{p}\,| = \mu r^2\,|\,\dot{\phi}\,|$



$$dA = \frac{1}{2} r^2 d\phi\,, \qquad \left(\frac{1}{2}\text{ base} \times \text{height of triangle}\right)$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{\ell}{2\mu} = \text{const.}$$

Radius vector sweeps out equal areas in equal times. Called *Kepler's Second Law*. This is also true in the CM frame, but the rate is different.

8.3 RUNGE-LENZ TREATMENT OF COULOMB FORCE

Let's examine the vectors which may be formed from $\vec{\ell}$, $\vec{p}$ and $\vec{r}$. The scalar products

$$\vec{r} \cdot \vec{\ell} = \vec{r} \cdot (\vec{r} \times \vec{p}) = 0, \tag{8.50}$$

$$\vec{p} \cdot \vec{\ell} = \vec{p} \cdot (\vec{r} \times \vec{p}) = 0. \tag{8.51}$$

vanish as pointed out before. The vector products are

$$\vec{M} \equiv \vec{\ell} \times \vec{r}\,, \tag{8.52}$$

$$\vec{\mathcal{L}} \equiv \vec{\ell} \times \vec{p}. \tag{8.53}$$

They are simply related:

$$\mu\,\dot{\vec{M}} = \mu\left(\underbrace{\dot{\vec{\ell}}}_{0} \times \vec{r} + \vec{\ell} \times \dot{\vec{r}}\right) = \vec{\ell} \times \vec{p} = \vec{\mathcal{L}}\,. \tag{8.54}$$

Next, consider

$$\dot{\vec{\mathcal{L}}} \equiv \vec{\ell} \times \dot{\vec{p}}. \tag{8.55}$$

Hamilton's equations for r_i, p_i are:

$$\dot{r}_i = \frac{\partial H_{rel}}{\partial p_i}, \text{ (identity)} \tag{8.56}$$

$$\dot{p}_i = -\frac{\partial H_{rel}}{\partial r_i} = -\frac{\partial U}{\partial r_i}, \tag{8.57}$$

But

$$\frac{\partial U(r)}{\partial r_i} = \frac{dU}{dr}\frac{\partial r}{\partial r_i}, \qquad r=\sqrt{r_1^2 + r_2^2 + r_3^2},$$

$$\frac{\partial r}{\partial r_i} = \frac{1}{2}\frac{2r_i}{\left(r_1^2 + r_2^2 + r_3^2\right)^{1/2}} = \frac{r_i}{r}, \tag{8.58}$$

so

$$\dot{\vec{\mathcal{L}}} = \vec{\ell} \times \dot{\vec{p}} = -\vec{\ell} \times \frac{\vec{r}}{r}\frac{dU}{dr},$$

or

$$\dot{\vec{\mathcal{L}}} = -\frac{\vec{M}}{r}\left(\frac{dU}{dr}\right). \tag{8.59}$$

Now

$$\vec{M} = \vec{\ell} \times \vec{r} = \left(\vec{r} \times \vec{p}\right) \times \vec{r} = \vec{r} \times \left(\vec{p} \times \vec{r}\right),$$

$$\left(\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)\right),$$

$$\Rightarrow \vec{M} = \vec{p}\left(\vec{r} \cdot \vec{r}\right) - \vec{r}\left(\vec{r} \cdot \vec{p}\right) = \vec{p}r^2 - \vec{r}\left(\vec{r} \cdot \vec{p}\right),$$

$$\vec{r} \cdot \vec{p} = r\left(\mu\dot{r}\right) = \mu r\dot{r}, \tag{8.60}$$

In addition,

$$\dot{\hat{e}}_r = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^2}\,\dot{r}\,,$$

$$= \frac{1}{r^2}\left(r\dot{\vec{r}} - \vec{r}\dot{r}\right). \tag{8.61}$$

($\hat{e}_r$ is unit vector along $\vec{r}$). This means

$$\vec{M} = \mu r^3 \dot{\hat{e}}_r\,, \tag{8.62}$$

and also that

$$\dot{\mathcal{L}} = -\mu r^2\, \dot{\hat{e}}_r \left(\frac{dU}{dr}\right) \tag{8.63}$$

All this has been done without reference to the form of U(r). Now assume

$$U(r) = -\frac{k}{r}\,, \qquad k = \underset{\substack{\uparrow \\ \text{Newton's gravitational} \\ \text{constant}}}{G}\, m_1 m_2$$

$$\Rightarrow \vec{F} = -\vec{\nabla}U = -\frac{k}{r^2}\,\hat{e}_r .$$

Then

$$\dot{\vec{\mathcal{L}}} = -\mu k\dot{\hat{e}}_r \qquad \text{or} \qquad \frac{d}{dt}\left(\vec{\mathcal{L}} + \mu k\hat{e}_r\right) = 0 . \tag{8.64}$$

A constant of the motion! This is the *Runge-Lenz* vector, $\vec{A}$.

$$\vec{A} \equiv \vec{\mathcal{L}} + \mu k\hat{e}_r = \left(\vec{r}\times\vec{p}\right)\times\vec{p} + \mu k\,\hat{e}_r .$$

Notice that $\frac{d\vec{A}}{dt} = 0$ only for $U = -\frac{k}{r}$. Can now show that

$$\vec{A}\cdot\vec{\ell} = \underbrace{\left[\left(\vec{r}\times\vec{p}\right)\times\vec{p}\right]}_{\perp \text{ to } \vec{r}\times\vec{p}}\cdot\underset{\substack{\uparrow \\ \vec{r}\times\vec{p}}}{\vec{\ell}} + \mu k\hat{e}_r\cdot\vec{\ell} = 0. \tag{8.65}$$

The vector $\vec{A}$ is in the plane of motion along with $\vec{r}$ and $\vec{p}$. Can get the equation of motion as follows.

$$\vec{A}\cdot\vec{r} = \vec{\mathcal{L}}\cdot\vec{r} + \mu k r ,$$

$$\vec{\mathcal{L}}\cdot\vec{r} = \left(\vec{\ell}\times\vec{p}\right)\cdot\vec{r} = \vec{\ell}\cdot\left(\vec{p}\times\vec{r}\right) = -\ell^2 ,$$

$$\therefore\ \vec{A}\cdot\vec{r} = -\ell^2 + \mu k r. \tag{8.66}$$

Let ϕ = angle between $-\vec{A}$ and $\vec{r}$. Then ($A \equiv |\vec{A}|$)

$$-Ar\cos\phi = -\ell^2 + \mu k r ,$$

$$\Rightarrow \frac{1}{r} = \frac{k\mu}{\ell^2}\left(1 + \frac{A}{\mu k}\cos\phi\right). \tag{8.67}$$

Define

$$\alpha \equiv \frac{\ell^2}{\mu k} ,$$

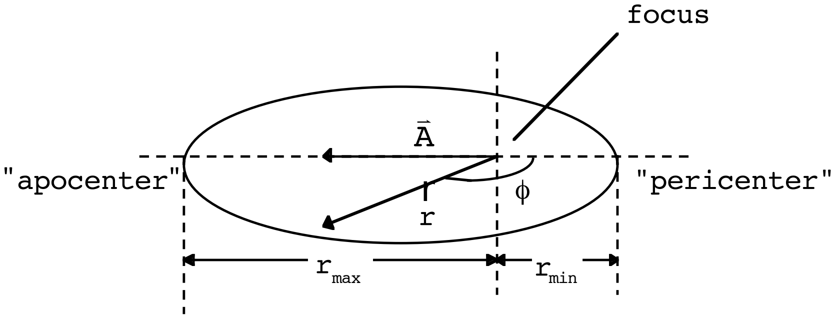
$$\varepsilon \equiv \frac{A}{\mu k}\text{. “eccentricity”} \tag{8.69}$$

Gives equation of *conic sections*. (Assume $\ell \neq 0$)

$$\left.\begin{array}{ll} \varepsilon > 1 & \text{hyperbola} \\ \varepsilon = 1 & \text{parabola} \end{array}\right\} \text{unbound motion}$$

$$\left.\begin{array}{ll} 0 < \varepsilon < 1 & \text{ellipse} \\ \varepsilon = 0 & \text{circle} \end{array}\right\} \text{bound}$$

Most important case: ellipse



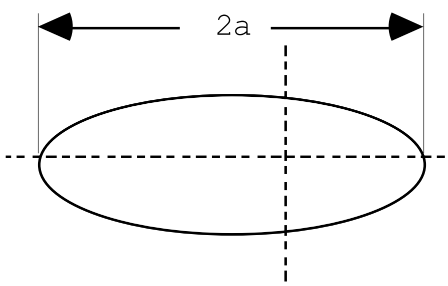
Notice when $\phi = \pi, 0$ (angle between $-\vec{A}$ & $\vec{r}$), $r = r_{max}, r_{min}$:

$$r_{max} = \frac{\alpha}{1 - \varepsilon}, \tag{8.70}$$

$$r_{min} = \frac{\alpha}{1 + \varepsilon}. \tag{8.71}$$

Thus $|\vec{A}|$ determines eccentricity, $\hat{A}$ (direction) determines direction of r_{max}.

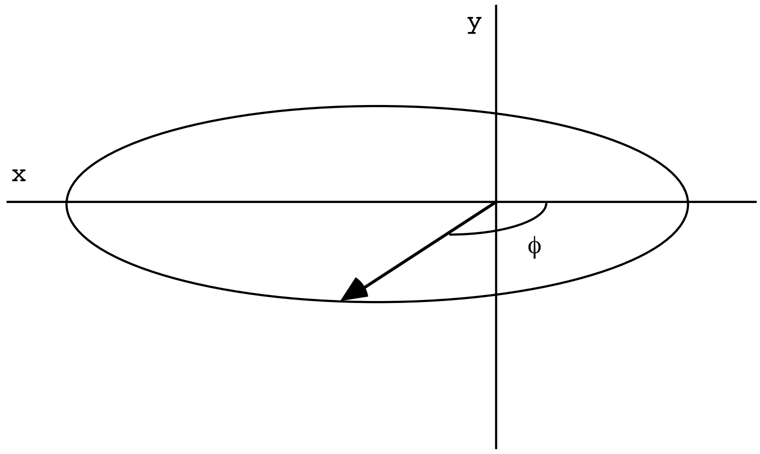
"Major axis", a:



$$a = \frac{1}{2}(r_{max} + r_{min}) = \frac{\alpha}{2}\left(\frac{1}{1-\varepsilon} + \frac{1}{1+\varepsilon}\right),$$

$$\Rightarrow a = \frac{\alpha}{2}\left(\frac{1-\varepsilon+1+\varepsilon}{1-\varepsilon^2}\right) = \frac{\alpha}{1-\varepsilon^2}. \tag{8.72}$$

Let's write the equation of motion in more recognizeable terms.



$$\frac{\alpha}{r} = 1 + \underbrace{\varepsilon\cos\phi}_{-\frac{x}{r}},$$

$$\Rightarrow \frac{1}{r}(\alpha+\varepsilon x) = 1,$$

$$\Rightarrow (\alpha+\varepsilon x)^2 = r^2 = (x^2+y^2),$$

$$\Rightarrow \alpha^2+\varepsilon^2x^2+2\alpha\varepsilon x - x^2 - y^2 = 0,$$

$$\Rightarrow x^2 - \frac{2\alpha\varepsilon}{1-\varepsilon^2}x + \frac{y^2}{1-\varepsilon^2} - \frac{\alpha^2}{1-\varepsilon^2} = 0,$$

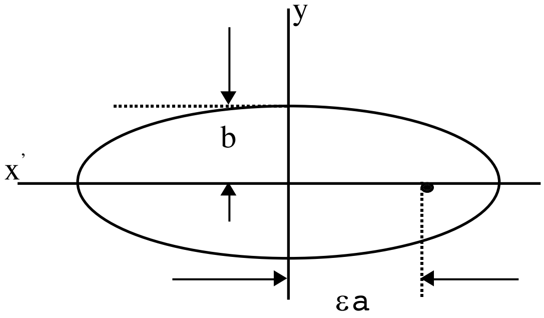
$$\Rightarrow (x-\varepsilon a)^2 + \frac{y^2}{1-\varepsilon^2} - \frac{\alpha^2}{1-\varepsilon^2} - \frac{\alpha^2\varepsilon^2}{(1-\varepsilon^2)^2} = 0,$$

$$\left(\frac{\alpha^2}{1-\varepsilon^2}\left(1+\frac{\varepsilon^2}{1-\varepsilon^2}\right) = \frac{\alpha^2}{(1-\varepsilon^2)^2} = a^2\right)$$

$$\Rightarrow (x-\varepsilon a)^2 + \frac{y^2}{1-\varepsilon^2} - a^2 = 0,$$

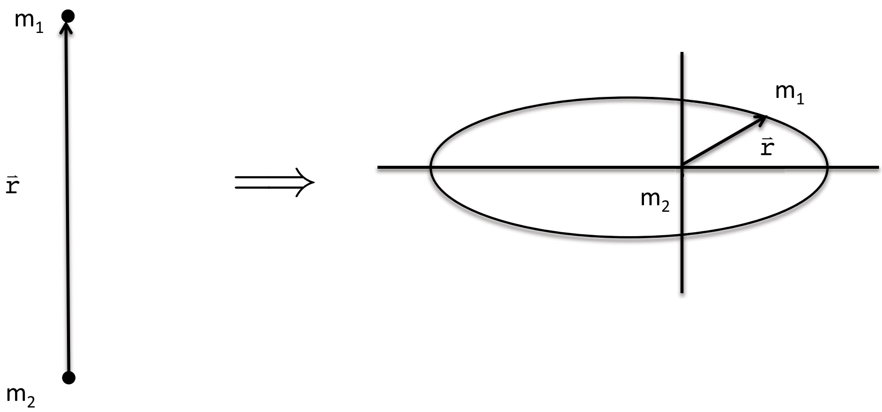
$$\Rightarrow \frac{(x-\varepsilon a)^2}{a^2} + \frac{y^2}{a^2(1-\varepsilon^2)} = 1.$$

Equation of an ellipse (if $0 < \varepsilon < 1$; just define $x' = x - \varepsilon a$). Can read off b, "minor axis":



$$b = a\sqrt{1-\varepsilon^2} = \frac{\alpha}{\sqrt{1-\varepsilon^2}}\,. \tag{8.73}$$

Have confirmed that equation of motion is an ellipse when $\vec{r}$ is plotted (Kepler's 1st law). What it represents:

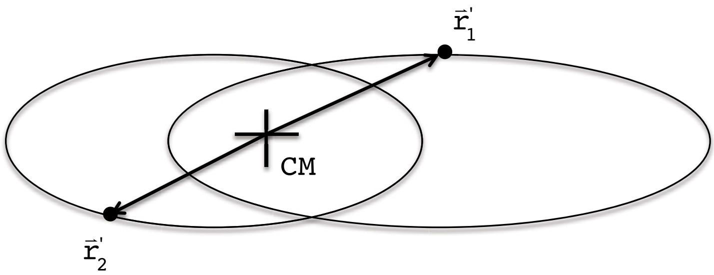


It is an ellipse when viewed from a coordinate system centered on m_1 or m_2 because

$$\vec{r}_1' = \frac{\mu}{m_1}\vec{r}, \tag{8.74}$$

$$\vec{r}_2' = -\frac{\mu}{m_2}\vec{r}, \tag{8.75}$$

when viewed from the CM, motions still form ellipses.



$\vec{r}_1'$, $\vec{r}_2'$ are measured from the CM, so the CM point is the common focus point for both ellipses. The eccentricity, ε, may be determined from the energy and the angular momentum as follows:

$$\vec{A}^2 = \left(\vec{\mathcal{L}} + \mu k \hat{e}_r\right)^2 = \vec{\mathcal{L}}^2 + \mu^2 k^2 + 2\mu k \vec{\mathcal{L}} \cdot \hat{e}_r\,,$$

$$\vec{\mathcal{L}}^2 = \left[(\vec{r} \times \vec{p}) \times \vec{p}\right]^2 = \left(\vec{\ell} \times \vec{p}\right) \cdot \left(\vec{\ell} \times \vec{p}\right) = \vec{\ell} \cdot \underbrace{\left(\vec{p} \times \left(\vec{\ell} \times \vec{p}\right)\right)}_{\vec{\ell}\,p^2 - \vec{p}\left(\vec{\ell} \cdot \vec{p}\right)},$$

$$\therefore \quad \vec{\mathcal{L}}^2 = \ell^2\, p^2\,.$$

From before $\left(\vec{\mathcal{L}} \cdot \vec{r} = -\ell^2\right)$:

$$\vec{\mathcal{L}} \cdot \hat{e}_r = -\frac{1}{r}\ell^2\,, \tag{8.76}$$

$$\vec{A}^2 = \ell^2 p^2 + \mu^2 k^2 - \frac{2\mu k}{r}\ell^2 = 2\mu\ell^2 \overbrace{\left(\frac{p^2}{2\mu} - \frac{k}{r}\right)}^{\equiv E} + \mu^2 k^2\,,$$

(E is the total internal energy of the system)

$$\Rightarrow \quad \vec{A}^2 = 2\mu\ell^2 E + \mu^2 k^2 , \tag{8.77}$$

$$\Rightarrow \quad \varepsilon = \frac{A}{\mu k} = \left(1 + \frac{2\ell^2 E}{\mu k^2}\right)^{1/2} . \tag{8.78}$$

Notice that E can be negative, zero, or positive. Again, for $\vec{\ell} \neq 0$:

$\varepsilon > 1$	hyperbola	$E > 0$
$\varepsilon = 1$	parabola	$E = 0$
$0 < \varepsilon < 1$	ellipse	$E < 0$
$\varepsilon = 0$	circle	$E = -\frac{\mu k^2}{2\ell^2} = -\frac{1}{2}\mu v^2$

(v = total velocity viewed from m_1 or m_2.)

8.4 LAGRANGIAN EQUATIONS OF MOTION

Let's do it again from a Lagrangian point of view to see the more standard treatment.

$$L = \frac{1}{2} M\dot{\vec{R}}^2 + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)}_{L_{rel}} . \tag{8.79}$$

Dependent variables: $\vec{R}(t)$, $\vec{r}(t)$:

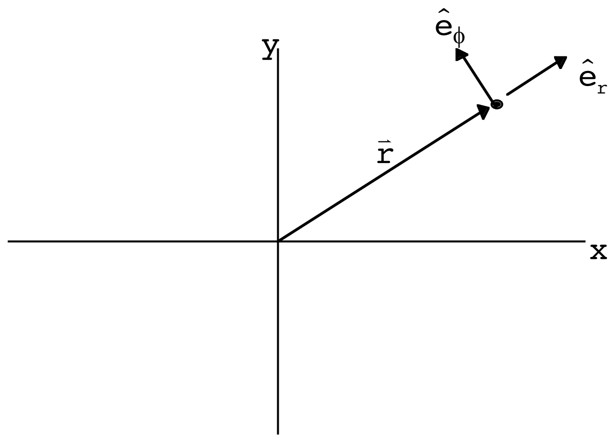
$$\vec{R} : \quad \frac{\partial L}{\partial \vec{R}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{R}}}\right) = 0 , \quad \Rightarrow \quad \ddot{\vec{R}} = 0 , \tag{8.80}$$

$$\vec{r} : \quad \frac{\partial L}{\partial \vec{r}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\right) = 0 , \quad \Rightarrow \quad \mu\ddot{\vec{r}} = -\frac{dU}{dr}\hat{e}_r , \tag{8.81}$$

$$\dot{\vec{\ell}} = \vec{r} \times \dot{\vec{p}} = \vec{r} \times \left(-\frac{dU}{dr}\hat{e}_r\right) = 0 . \tag{8.82}$$

$\Rightarrow$ Motion is in a plane, as before. Pick $\vec{\ell}$ along z:

(cylindrical coordinates)



$$\ddot{\vec{r}} = \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r + \dot{r}\dot{\phi}\,\hat{e}_\phi + r\ddot{\phi}\,\hat{e}_\phi + r\dot{\phi}\,\dot{\hat{e}}_\phi. \tag{8.83}$$

We have (Ch.2):

$$\dot{\hat{e}}_r = \dot{\phi}\hat{e}_\phi\,, \tag{8.84}$$

$$\dot{\hat{e}}_\phi = -\dot{\phi}\hat{e}_r\,. \tag{8.85}$$

$$\Rightarrow \ddot{\vec{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\hat{e}_\phi\,. \tag{8.86}$$

Substitute into (8.81) above:

$$\hat{e}_r : \quad \mu\left(\ddot{r} - r\dot{\phi}^2\right) = -\frac{dU}{dr}\,, \tag{8.87}$$

$$\hat{e}_\phi : \quad 2\dot{r}\dot{\phi} + r\ddot{\phi} = 0\,. \tag{8.88}$$

These are the $\text{eq}^{\underline{n}}\text{s}$ of motion. Then,

$$(8.88) \Rightarrow 2\frac{\dot{r}}{r} = -\frac{\ddot{\phi}}{\dot{\phi}} \Rightarrow 2\ln\left(\frac{r}{c_1}\right) = -\ln\left(\frac{\dot{\phi}}{c_2}\right),$$

$$\Rightarrow \left(\frac{r}{c_1}\right)^2 = \left(\frac{c_2}{\dot{\phi}}\right) \Rightarrow r^2\dot{\phi} = \text{const.}\left(= \frac{\ell}{\mu}\right). \tag{8.89}$$

This is Kepler's second law, derived again. Also

$$(8.87) \Rightarrow \quad \ddot{r} - \frac{\ell^2}{\mu^2 r^3} = -\frac{1}{\mu}\frac{dU}{dr}\,. \tag{8.90}$$

Solving this nonlinear differential eq$^{\underline{n}}$ gives r(t). Can derive a different for r(ϕ) as follows.

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{l}{\mu r^2}\frac{dr}{d\phi} = \frac{l}{\mu r^2}(-r^2)\underbrace{\left(-\frac{1}{r^2}\frac{dr}{d\phi}\right)}_{\frac{d\left(\frac{1}{r}\right)}{d\phi}},$$

$$\Rightarrow \dot r = -\frac{\ell}{\mu}\frac{d\left(\frac{1}{r}\right)}{d\phi}. \tag{8.91}$$

Likewise

$$\dot\phi = \frac{\ell}{\mu}\frac{1}{r^2},$$

$$\frac{d^2 r}{dt^2} = -\frac{\ell}{\mu}\frac{d\phi}{dt}\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2} = -\frac{\ell}{\mu}\left(\frac{\ell}{\mu}\frac{1}{r^2}\right)\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2},$$

$$\Rightarrow \frac{d^2 r}{dt^2} = -\frac{\ell^2}{\mu^2}\frac{1}{r^2}\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}. \tag{8.92}$$

Substitute into (8.90) above:

$$-\frac{\ell^2}{\mu^2}\frac{1}{r^2}\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}-\frac{\ell^2}{\mu^2}\frac{1}{r^3}=-\frac{1}{\mu}\frac{dU}{dr},$$

or

$$\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}+\frac{1}{r}=\frac{\mu}{\ell^2}r^2\frac{dU}{dr}. \tag{8.93}$$

Given U(r), can find r(ϕ). Conversely, can find U(r) given some r(ϕ).

Example 1:

$$U(r) = -\frac{k}{r}. \quad \text{Define} \quad u \equiv \frac{1}{r}$$

$$\frac{d^2u}{d\phi^2}+u=\frac{k\mu}{\ell^2}. \tag{8.94}$$

A simple linear nonhomogeneous diff. eq^n. Same as for Hooke's law oscillator with a constant force.

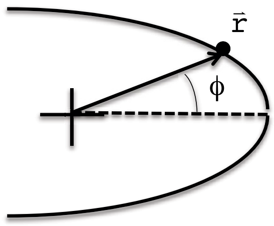
Homogeneous solution: $C\cos(\phi - \phi 1)$

Particular solution: $\frac{k\mu}{\ell^2}$ (like const. displacemenr of spring for const. force)

General solution:

$$u=\overset{\overset{\frac{1}{\alpha}}{\downarrow}}{\frac{k\mu}{\ell^2}}\left(1+\overset{\overset{\text{unknown}}{\downarrow}}{C}\cos\left(\phi-\phi_1\right)\right) \tag{8.95}$$

Eq^n of conic sections again. Supply the initial conditions. Let's agree to measure:



(ϕ measured from pericenter)

Value of C?

$$\frac{1}{r_{min}} = \frac{1}{\alpha}\left(1 + C\cos\phi_1\right),$$

$$\Rightarrow C\cos\phi_1 = \left(\frac{\alpha}{r_{min}} - 1\right). \tag{8.96}$$

Taking derivative of $u = \frac{1}{\alpha}\left(1 + C\ \cos(\phi - \phi_1)\right)$, we get

$$-\frac{1}{r^2}\dot{r} = -\frac{C}{\alpha}\sin(\phi - \phi_1)\frac{d\phi}{dt}.$$

Because $\dot{r} = 0$ at $r = r_{min}$, and assuming $C \neq 0$, we get

$$0 = -\frac{C}{\alpha}\sin\phi_1\frac{d\phi}{dt} \Rightarrow \phi_1 = 0,$$

$$\Rightarrow \quad C = \frac{\alpha}{r_{min}} - 1. \tag{8.97}$$

From previous solution we know that $r_{min} = \dfrac{\alpha}{1 + \varepsilon}$:

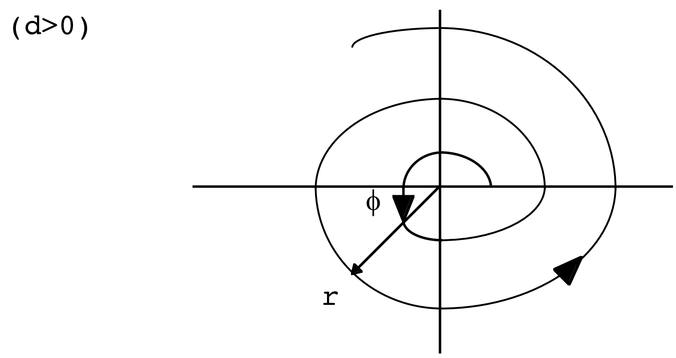
$$\Rightarrow C = \frac{\alpha}{\alpha}(1 + \varepsilon) - 1 = \varepsilon,$$

$$\Rightarrow \frac{1}{r} = \frac{1}{\alpha}\left(1 + \varepsilon\cos\phi\right). \tag{8.98}$$

[Note: Can also get r_{max} or r_{min} directly from $E = \dfrac{p^2}{2\mu} - \dfrac{k}{r_{min}}$, $p^2 = \mu^2 r_{min}^2 \dot{\phi}_{min}^2$, $\dot{\phi}_{min} = \dfrac{\ell}{\mu r_{min}^2}$ and then solving for r from the resulting quadratic $eq^{\underline{n}}$. See prob. 8.15.]

Example 2: $r = \frac{1}{c}e^{d\phi}$, $u = ce^{-d\phi}$ $(c > 0)$

Exponential spiral:



Force law?

$$\frac{d^2u}{d\phi^2} = cd^2e^{-d\phi} = d^2u,$$

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu r^2}{\ell^2}\frac{dU}{dr},$$

$$\Rightarrow \frac{1}{r}\left(d^2 + 1\right) = \frac{\mu r^2}{\ell^2}\frac{dU}{dr},$$

$$\Rightarrow F(r) = -\frac{dU}{dr} = -\frac{\ell^2}{\mu r^3}\left(d^2 + 1\right). \text{ (attractive)} \tag{8.99}$$

Inverse *cube* law. This is not the only type of motion in such a force field. This is a special type of solution of the inverse cube law corresponding to E = 0.

8.5 CELESTIAL MECHANICS

Let's derive some other useful results for the inverse square law, $U(r) = -\frac{k}{r}$.

$$\ddot{r} = \frac{\ell^2}{\mu^2 r^3} - \frac{1}{\mu}\frac{dU}{dr}, \tag{8.100}$$

$$\int dt\left(\ddot{r}\dot{r} - \frac{\ell^2}{\mu^2}\frac{\dot{r}}{r^3} + \frac{1}{\mu}\frac{dU}{dr}\dot{r}\right) = 0\,,$$

$$\Rightarrow \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = E. \qquad (\text{const.}) \tag{8.101}$$

This is just the statement of energy conservation since

$$\dot{r}^2 + \frac{\ell^2}{\mu^2 r^2} = \dot{r}^2 + r^2\dot{\phi}^2 = v^2,$$

$$\Rightarrow \quad E = \frac{1}{2}\mu v^2 + U(r)\,,$$

$$\Rightarrow \quad v^2 = \frac{2}{\mu}\left(E + \frac{k}{r}\right). \tag{8.102}$$

E is related to "a" by

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{\alpha}{1-\left(1+\frac{2\ell^2 E}{\mu k^2}\right)} = -\frac{\alpha\mu k^2}{2\ell^2 E}\,,$$

$$\Rightarrow \qquad a = -\frac{k}{2E} > 0\,,$$

$$\uparrow$$

$$\text{since } E < 0 \text{ for an ellipse}$$

$$\Rightarrow v^2 = \frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right), \tag{8.103}$$

for elliptical motion.

There is a connection between τ, the period of elliptical motion, and a, the semi-major axis, that is well known for $U(r) = -\frac{k}{r}$. The connection is stated roughly as

$$\underset{\text{“proportional to”}}{\tau^2 \underset{\uparrow}{\propto} a^3}$$

and is known as *Kepler's third law*. We actually derived this rough form last semester from an invariance argument. Recall that

$$L_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - U(r). \tag{8.104}$$

with $U(r) = c\, r^n$ under the transformation

$$\begin{cases} \vec{r} \to \alpha\vec{r}, \\ t \to \alpha^{1-n/2}\, t, \end{cases} \tag{8.105}$$

behaves as

$$L_{rel} \to \alpha^n L_{rel}.$$

$\Rightarrow$ Eq^n of motion from $\delta\int L_{rel}\, dt = 0$ unchanged. For the inverse square force law

$$U(r) = -\frac{k}{r} \quad , \quad n = -1,$$

$$\Rightarrow \quad \frac{\tau'}{\tau} = \alpha^{3/2} = \left(\frac{a'}{a}\right)^{3/2},$$

$$\Rightarrow \qquad \tau^2 \propto a^3.$$

as we had before. To get the full form of Kepler's 3rd law consider

$$\tau = \int_{\substack{\text{full}\\ \text{orbit}}} \frac{rd\phi}{v_\phi} = 2\int_0^\pi \frac{r(\phi)}{v_\phi(\phi)}\, d\phi, \tag{8.106}$$

$$\vec{\ell} = \vec{r}\times\vec{p} = \vec{r}\times\vec{p}_\phi \Rightarrow \ell = rp_\phi = r\mu v_\phi,$$

$$v_\phi = \frac{\ell}{\mu r},$$

$$\tau = 2\frac{\mu}{\ell}\int_0^\pi d\phi\, r^2(\phi),$$

$$r^2 = \left(\frac{\ell^2}{\mu k}\right)^2 \frac{1}{(1 + \varepsilon \cos \phi)^2},$$

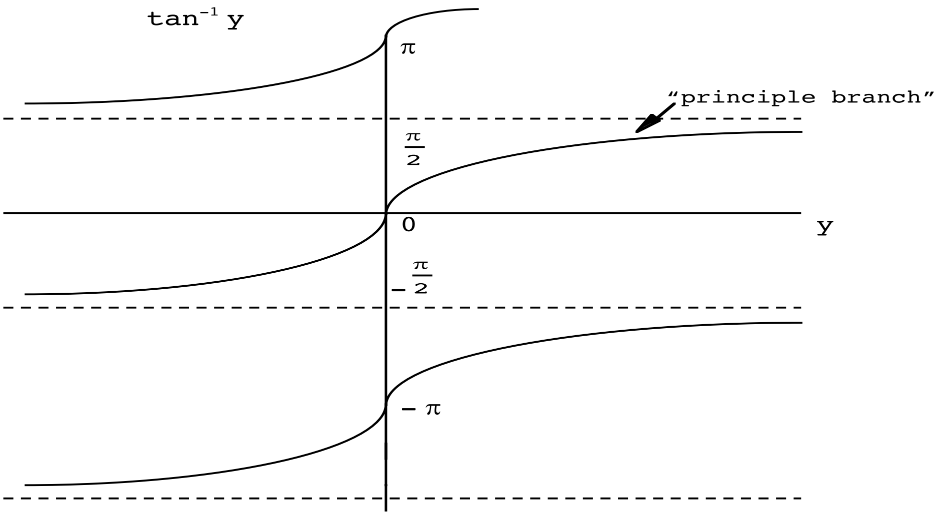
$$\Rightarrow \tau = \underbrace{2\frac{\mu}{\ell}\frac{\ell^4}{\mu^2 k^2}}_{2\left(\frac{\mu\alpha^3}{k}\right)^{1/2}} \int_0^{\pi} \frac{d\phi}{(1+\varepsilon \cos\phi)^2},$$

so

$$\tau = 2\left(\frac{\mu\alpha^3}{k}\right)^{1/2} \int_0^{\pi} \frac{d\phi}{(1+\varepsilon \cos\phi)^2}. \tag{8.107}$$

Useful integral:

$$\int \frac{d\phi}{(1+\varepsilon \cos\phi)^2} = \frac{\varepsilon \sin \phi}{\left(\varepsilon^2 - 1\right)\left(1 + \varepsilon \cos \phi\right)} + \frac{2}{\left(1 - \varepsilon^2\right)^{3/2}} \tan^{-1}\left(\frac{(1 - \varepsilon) \tan \frac{\phi}{2}}{\sqrt{1 - \varepsilon^2}}\right).$$



Plugging in the limits, $\phi = 0, \pi$, we get

$$\tau = 2\left(\frac{\mu\alpha^3}{k}\right)^{1/2}\left[\frac{2}{\left(1-\varepsilon^2\right)^{3/2}}\right]\underbrace{\left(\tan^{-1}(\infty)-\tan^{-1}(0)\right)}_{\frac{\pi}{2}\left(\begin{array}{l}\text{stay on}\\ \text{a single brach}\end{array}\right)},$$

$$\tau = 2\pi\left(\frac{\mu}{k}\right)^{1/2}\left(\frac{\alpha}{1-\varepsilon^2}\right)^{3/2},$$

$$\Rightarrow \quad \tau^2 = \left(\frac{4\pi^2\mu}{k}\right)a^3. \tag{8.108}$$

Kepler's feelings about discovering his $3^{\underline{\text{rd}}}$ law:

"Now, since the dawn eight months ago, since the broad daylight three months ago, and since a few days ago, when the full sun illuminated my wonderful speculations, nothing holds me back. I yield freely to the sacred frenzy; I dare frankly to confess that I have stolen the golden vessels of the Egyptians to build a tabernacle for my God far from the bounds of Egypt. If you pardon me, I shall rejoice; if you reproach me, I shall endure. The die is cast, and I am writing the book – to be read either now or by posterity, it matters not. It can wait a century for a reader, as God himself has waited six thousand years for a witness."

From "The Discoverers" by D. Boorstin

Let's do the above integration from $\phi = 0$ to an arbitrary angle ϕ:

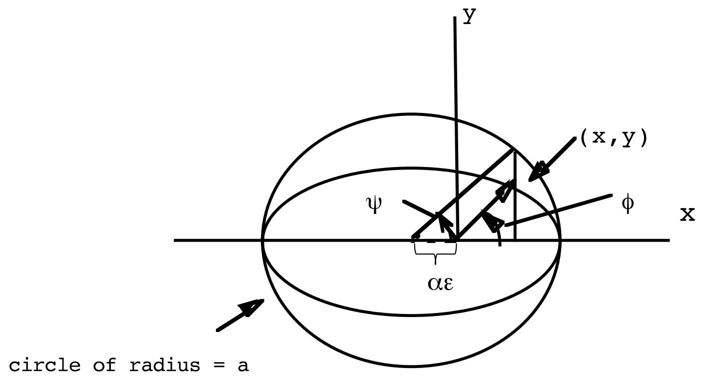
$$t = \left(\frac{\mu\alpha^3}{k}\right)^{1/2} \int_0^{\phi} \frac{d\phi'}{(1 + \varepsilon \cos\phi')^2},$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{“mean anomaly”}}}{\frac{2\pi t}{\tau}} = \left(1 - \varepsilon^2\right)^{3/2} \int_0^{\phi} \frac{d\phi'}{(1 + \varepsilon \cos\phi')^2}. \tag{8.109}$$

Using the previous integral gives:

$$\frac{2\pi t}{\tau} = 2 \tan^{-1}\left(\frac{(1 - \varepsilon) \tan \frac{\phi}{2}}{\sqrt{1 - \varepsilon^2}}\right) - \frac{\varepsilon\sqrt{1 - \varepsilon^2} \sin \phi}{(1 + \varepsilon \cos \phi)}. \tag{8.110}$$

Careful use of this equation gives $t(\phi)$. But what we usually want is $\phi(t)$! The need to invert this relationship gives rise to *Kepler's equation*. We can derive it as follows. Adopt the following coordinate system:



ellipse: $\dfrac{(x + \varepsilon a)^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

$\psi \equiv$ “eccentric anomaly”.

First, get relationship between ϕ and ψ.

$$\cos \psi = \frac{x + a\varepsilon}{a} \Rightarrow x = a(\cos \psi - \varepsilon), \tag{8.111}$$

$$\sin \psi = \frac{\sqrt{a^2 - (x + a\varepsilon)^2}}{a} = \frac{y \frac{a}{b}}{a} = \frac{y}{b} \Rightarrow y = b \sin \psi,$$

or

$$y = a\sqrt{1-\varepsilon^2}\,\sin\psi\,, \tag{8.112}$$

so

$$r^2 = x^2 + y^2 = a^2(1-\varepsilon\cos\psi)^2\,,$$

$$\Rightarrow r = a(1-\varepsilon\cos\psi)\,. \tag{8.113}$$

Remember:

$$\frac{1}{r} = \frac{1}{\alpha}(1+\varepsilon\cos\phi)$$

$$\Rightarrow 1+\varepsilon\cos\phi = \frac{\alpha}{a}\frac{1}{(1-\varepsilon\cos\psi)}\,,\ \frac{\alpha}{a} = 1-\varepsilon^2. \tag{8.114}$$

Taking the differential of both sides of this gives

$$d(1+\varepsilon\cos\phi) = (1-\varepsilon^2)\,d\left(\frac{1}{1-\varepsilon\cos\psi}\right),$$

$$\Rightarrow d\phi = (1-\varepsilon^2)\,\frac{\sin\psi}{\sin\phi}\,\frac{d\psi}{(1-\varepsilon\cos\psi)^2}\,. \tag{8.115}$$

Moreover, by definition

$$\sin\phi = \frac{y}{r} = \frac{b\sin\psi}{a(1-\varepsilon\cos\psi)}\,,$$

$$\Rightarrow\ d\phi = \sqrt{1-\varepsilon^2}\,\frac{d\psi}{1-\varepsilon\cos\psi}\,. \tag{8.116}$$

We now have

$$\frac{d\phi}{(1+\varepsilon\cos\phi)^2} \underset{\uparrow}{=} \left(\sqrt{1-\varepsilon^2}\,\frac{d\psi}{1-\varepsilon\cos\psi}\right)\left(\frac{1}{(1-\varepsilon^2)^2}\,(1-\varepsilon\cos\psi)^2\right),$$

use (8.114) and (8.116)

$$= \frac{1}{(1-\varepsilon^2)^{3/2}}\,(1-\varepsilon\cos\psi)\,d\psi\,, \tag{8.117}$$

so now

$$\frac{2\pi t}{\tau} = \left(1-\varepsilon^2\right)^{3/2}\int_0^{\phi}\frac{d\phi'}{\left(1+\varepsilon\cos\phi'\right)^2},$$

$$\frac{2\pi t}{\tau} = \int_0^{\psi}(1-\varepsilon\cos\psi')d\psi' = \psi - \varepsilon\sin\psi\,. \tag{8.118}$$

Eq$^{\underline{n}}$ (8.118) is Kepler's equation. One can also show (integrate (8.116)) that

$$\tan\frac{\phi}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\psi}{2}\,. \tag{8.119}$$

So, the method of finding $\phi(t)$ using Kepler's eq$^{\underline{n}}$ is:

1. Solve Kepler's eq$^{\underline{n}}$ approximately for $\psi(t)$.
2. Use (8.99) to find $\phi(t)$ from

$$\phi(t) = 2\tan^{-1}\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\psi(t)}{2}\right).$$

8.6 GENERAL RELATIVITY MODIFICATION

I now want to talk a little about a small modification to these orbits caused by Einstein's general theory of relativity. Start with the effective (approx.) potential for perihelion (Mercury) precession. I am leaving out terms in effective potential which do not cause precession. This potential is observer dependent. The effective potential with respect to an observer at r = ∞:

$$U_{\text{eff}}^{GR}(r) = -\frac{GMm}{r} - \frac{3}{mc^2}\left(\frac{GMm}{r}\right)^2 . \tag{8.120}$$

This is not the same as the effective potential as, for example in Marion, which is with respect to the observer in orbit.

Let's put this into our orbit equation:

$$\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2} + \frac{1}{r} = \frac{\mu r^2}{\ell^2}\,\frac{dU_{\text{eff}}}{dr} .$$

Let $u = \frac{1}{r}$,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{\ell^2}\left(\underbrace{GMm}_{=k} + \frac{6}{mc^2}\,(GMm)^2 u\right) . \tag{8.121}$$

Solve:

$$\frac{d^2u}{d\phi^2} + (1-\delta)\,u = \frac{\mu k}{\ell^2} , \tag{8.122}$$

$$\delta = \frac{6G^2M^2m\mu}{\ell^2c^2} . \tag{8.123}$$

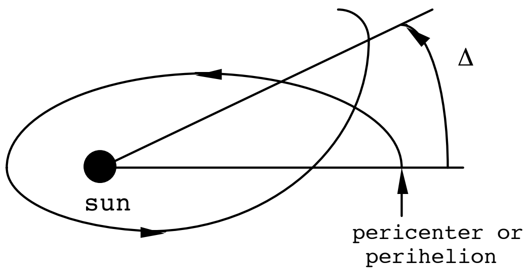
Solution: $u = \underbrace{\frac{k\mu}{(1-\delta)\,\ell^2}}_{=\frac{1}{\alpha(1-\delta)}} \left(1 + \underset{\substack{\uparrow \\ \text{analog of eccenticity} \\ \text{constant}}}{\varepsilon'} \cos\left(\sqrt{1-\delta}\,\phi\right)\right) .$ (8.124)

Most important change:

$$\phi = 0 \quad : \quad \frac{1}{r} = \frac{1}{r_{\min}}$$

$$\phi = \frac{2\pi}{\sqrt{1-\delta}} \quad : \quad \frac{1}{r} = \frac{1}{r_{\min}} \quad \text{again.}$$

Notice $\dfrac{2\pi}{\sqrt{1 - \delta}} > 2\pi \quad \Rightarrow$ orbit precesses positively $(\delta > 0)$.



$$\frac{2\pi}{\sqrt{1 - \delta}} \simeq 2\pi\left(1 + \frac{\delta}{2}\right) \equiv 2\pi + \Delta, \tag{8.125}$$

$$\Rightarrow \quad \Delta = \pi\delta = \frac{6\pi G^2 M^2 m\mu}{\ell^2 c^2} \tag{8.126}$$

From α $\text{def}^{\underline{n}}$:

$$\ell^2 = \alpha\mu k = a\left(1 - \varepsilon^2\right)\mu k = a\left(1 - \varepsilon^2\right) GMm\mu,$$

$$\Rightarrow \Delta = \frac{6\pi GM}{ac^2\left(1 - \varepsilon^2\right)} = \frac{6\pi GM}{\alpha c^2}. \tag{8.127}$$

Amounts to ~43 sec. of arc/century for Mercury.

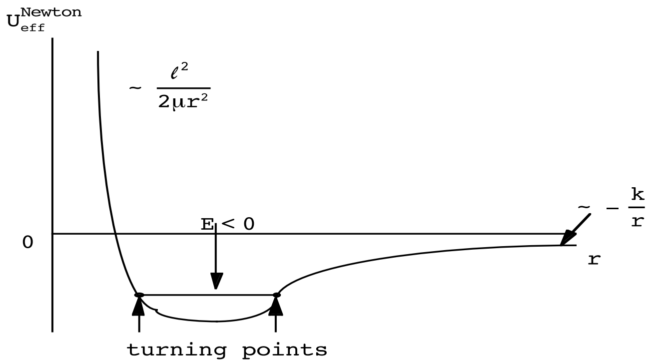
Newtonian effective potential:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{=U_{eff}^{Newton}(r)}. \tag{8.128}$$

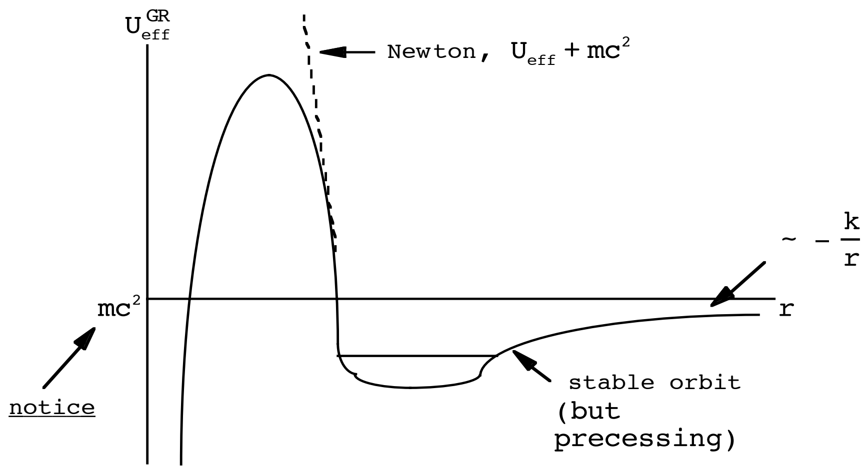
Define,

$$F_{centrifugal} = -\frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2}\right) = \frac{\ell^2}{\mu r^3}. \tag{8.129}$$

Looks like:



The full general relativity effective potential including centrifugal term:



New G.R. effect: *capture* (realized in "black holes")

8.7 ORBITAL STABILITY

Equation which describes orbital motion (from (8.90)):

$$\ddot{r} - \frac{\ell^2}{\mu^2 r^3} = -\frac{1}{\mu}\frac{dU}{dr} \quad . \tag{8.130}$$

Let r_0 be the radius of a nearly circular orbit, and let $r = r_0 + x$, where $\frac{x}{r_0} << 1$. For a circular orbit $\left(f \equiv \frac{F}{\mu}\right)$

$$\Rightarrow \qquad r_0^3 = -\frac{\left(\ell/\mu\right)^2}{f(r_0)} \, . \tag{8.131}$$

Then

$$\ddot{x} - \left(\frac{\ell}{\mu}\right)^2 \left(r_0 + x\right)^{-3} = f\left(r_0 + x\right) . \tag{8.132}$$

Assume

$$f\left(r_0 + x\right) = f(r_0) + \underbrace{\left(r - r_0\right)}_{x} f'\left(r_0\right) + \ldots \quad . \tag{8.133}$$

Also

$$\left(r_0 + x\right)^{-3} = r_0^{-3}\left(1 + \frac{x}{r_0}\right)^{-3} = r_0^{-3}\left(1 - \frac{3x}{r_0}\right) . \tag{8.134}$$

Then

$$\ddot{x} - \left(\frac{\ell}{\mu}\right)^2 \underset{\substack{\uparrow \\ \text{substitute the above}}}{r_0^{-3}} \left(1 - \frac{3x}{r_0}\right) = f\left(r_0\right) + x f'\left(r_0\right) . \tag{8.135}$$

$$\Rightarrow \ \ddot{x} + \frac{\left(\frac{\ell}{\mu}\right)^2 f(r_0)}{\left(\frac{\ell}{\mu}\right)^2}\left(1 - \frac{3x}{r_0}\right) = f(r_0) + x f'(r_0) , \tag{8.136}$$

Let

$$\alpha^2 \equiv -\frac{3f\left(r_0\right)}{r_0} - f'\left(r_0\right) . \tag{8.137}$$

Then

$$\alpha^2 > 0 : x = c_1 \sin \alpha t + c_2 \cos \alpha t - \underline{\underline{\text{stable}}} \tag{8.138}$$

$$\left.\begin{aligned} \alpha^2 = 0 &: x = c_1 t + c_2 \\ \alpha^2 < 0 &: x = c_1 e^{|\alpha| t} + c_2 e^{-|\alpha| t} \end{aligned}\right\} \underline{\underline{\text{unstable}}} \tag{8.139}$$

Condition for stability:

$$\frac{3f(r_0)}{r_0} + f'(r_0) < 0,$$

or (notice f<0)

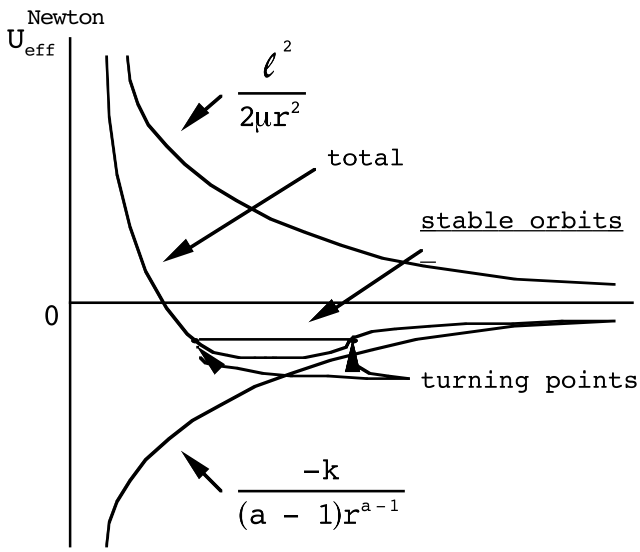
$$\Rightarrow r_0 \frac{F'(r_0)}{F(r_0)} > -3. \tag{8.140}$$

Try this out for $F(r) = -\dfrac{k}{r^a}$ $\left(U = -\dfrac{k}{(a-1)\, r^{a-1}}\right)$,

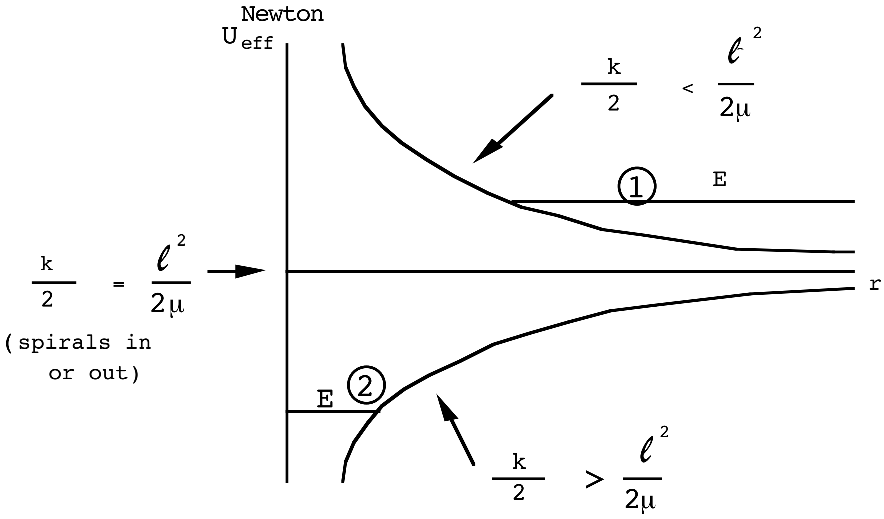
$$F'(r) = \frac{ak}{r^{a+1}}, \quad \frac{F'(r_0)}{F(r_0)} = -\frac{a}{r_0}.$$

$\Rightarrow$ stable if -a > -3 or a < 3. (Must also have a > 1 for as $U \to 0$ as $r \to \infty$.) Easily interpretable (k > 0 for all cases):

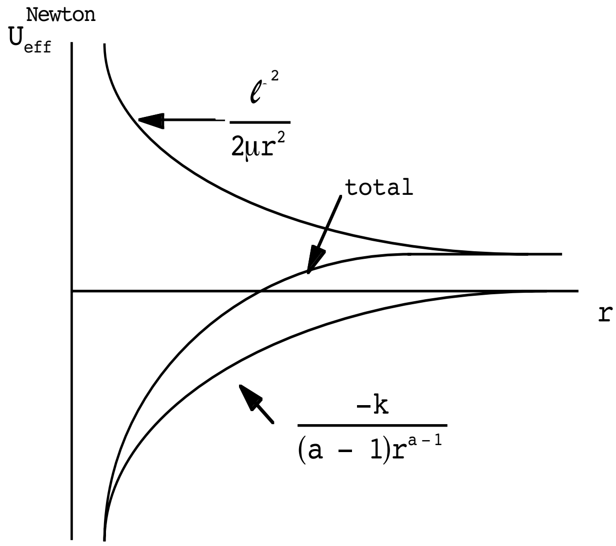
a < 3 case:



a = 3 case:



a > 3 case:



8.8 VIRIAL THEOREM

Now consider (τ = period of motion)

$$\left\langle \frac{d}{dt}\left(f(r)\,\dot{r}\right)\right\rangle_t \equiv \frac{1}{\tau}\int_0^\tau \frac{d}{dt}\left(f(r)\,\dot{r}\right)dt\,, \tag{8.141}$$

$$= \frac{1}{\tau}\left[f(r)\dot{r}\right]\Big|_0^\tau = 0\,.$$

for a periodic system. (Even if it is not strictly periodic, can let $\tau \to \infty$ and make or assume

$$\lim_{\tau\to\infty}\frac{1}{\tau}\left[f(r)\,\dot r\right]\Big|_0^{\tau} = 0. \tag{8.142}$$

if the system is *bounded*.) Now we have

$$\frac{d}{dt}\left(f(r)\,\dot r\right) = f'(r)\dot r^2 + f(r)\ddot r\,, \tag{8.143}$$

$$\Rightarrow \langle f'(r)\dot r^2\rangle + \langle f(r)\ddot r\rangle = 0. \text{ (averaged over time)} \tag{8.144}$$

Can now use

$$\ddot r - \frac{\ell^2}{\mu^2 r^3} = -\frac{1}{\mu}\frac{dU}{dr}\,,$$

so

$$\left\langle f'(r)\,\dot r^2 + \frac{\ell^2}{\mu^2}\frac{f(r)}{r^3}\right\rangle = \frac{1}{\mu}\left\langle f(r)\,\frac{dU}{dr}\right\rangle. \tag{8.145}$$

First example: $f(r) = \text{const.}$

$$\Rightarrow \quad \langle \ddot{r} \rangle = 0\,,$$

$$\Rightarrow \quad \frac{\ell^2}{\mu}\left\langle \frac{1}{r^3} \right\rangle = \left\langle \frac{dU}{dr} \right\rangle = -\left\langle F(r) \right\rangle, \tag{8.146}$$

or, in words:

Average centrifugal "force" = Average radial force

Must be true for any periodic system. Another example: $f(r) = \frac{1}{2}\mu r$

$$\left\langle \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} \right\rangle = \frac{1}{2}\left\langle r\,\frac{dU}{dr} \right\rangle,$$

$$\langle T \rangle = \frac{1}{2}\left\langle r\,\frac{dU}{dr} \right\rangle. \tag{8.147}$$

This is just the "virial theorem" applied to central-force motion. Let's take $U(r) = -\frac{k}{r}$. Then

$$\langle T \rangle = \frac{1}{2}\left\langle \frac{k}{r} \right\rangle.$$

$$\Rightarrow \quad \langle T \rangle = -\frac{1}{2}\langle U \rangle\,. \tag{8.148}$$

But, $T + U = E \Rightarrow \langle T \rangle + \langle U \rangle = E$,

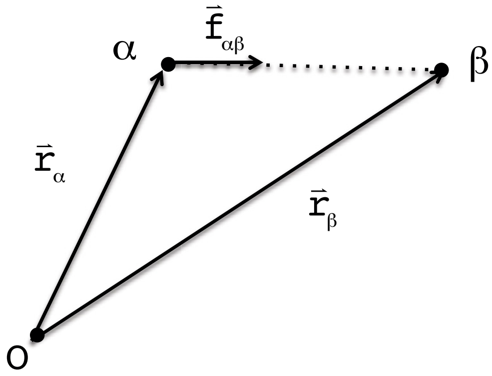
$$\Rightarrow \begin{cases} \langle U \rangle = 2E & < 0 \\ \langle T \rangle = -E & > 0 \end{cases} \quad \text{for bounded motion} \tag{8.149}$$

(For circular orbits, can take off $\langle\ \rangle$.)

Application: galaxy clusters $\Rightarrow$ must be "dark matter" in the universe. Outweighs visible matter by a factor of 5!

8.9 PROBLEMS

1. A system of point particles interact via forces which follow the "strong form" of Newton's second law. That is, the force of β on α, $\vec{f}_{\alpha\beta}$, points along the instantaneous line connecting them, as shown.



Given the usual connection between fixed ($\vec{r}_\alpha$) and center of mass ($\vec{r}_\alpha$) coordinates,

$$\vec{r}_\alpha = \vec{r}_\alpha^{\,'} + \vec{R},$$

($\vec{R} = \frac{1}{M}\sum_\alpha \vec{r}_\alpha$, M is total mass) and the total force on α,

$$\dot{\vec{p}}_\alpha = \underbrace{\vec{F}_\alpha^{(e)}}_{\left(\substack{\text{external}\\ \text{force on }\alpha}\right)} + \underbrace{\sum_\beta \vec{f}_{\alpha\beta}}_{\left(\substack{\text{internal}\\ \text{force}}\right)},$$

Show that the total torque, $\dot{\vec{L}} = \sum_\alpha \dot{\vec{L}}_\alpha$ ($\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$), for an external force of the form,

$$\vec{F}_\alpha^{(e)} = m_\alpha \vec{g},$$

is simply given by

$$\dot{\vec{L}} = \vec{R} \times \vec{F}^{(e)},$$

where $\vec{F}^{(e)} = \sum_\alpha \vec{F}_\alpha^{(e)}$ is the total external force.

2. Show that

$$\frac{d^2\vec{\mathcal{L}}}{dt^2} = -\frac{k}{\mu}\vec{\mathcal{L}}, \quad (\vec{\mathcal{L}} = \vec{\ell} \times \vec{p})$$

when the potential $U(r) = \frac{1}{2}kr^2$. Comment on the expected solution.

3. Investigate pure radial bound motion $(\ell = 0, E<0)$ in an inverse square force field, $F(r) = -\frac{k}{r^2}$, for two point masses. Let "h" be the maximum amplitude for pure radial motion. Imagine the particles move through one another as they pass. How is the period of pure radial motion (τ_{rad}) related to the period of a circular orbit of readius r (τ_{orbit}) when h = r?

4. As a follow-on to prob. 3, show that the equation of bound radial motion is a cycloid in time; that is,

$$r = \frac{h}{2}(1 - \cos\phi), \quad t = \frac{h}{2}\sqrt{\frac{\mu h}{2k}}(\phi - \sin\phi),$$

where h is the maximum radial distance between the masses. What is the relationship between r and t when E = 0? (E = total energy.) [Hint: Try making a change of variable, $r = \frac{h}{2}(1-\cos\phi)$, directly in the r integral.]

5. You are given that the distance of separation of two planets is given by

$$r = d(1 + \sin\phi),$$

where $d > 0$ is a constant and ϕ is the angle. (Ignore the fact that the planets strike one another at $\theta = 270^\circ$.)

a) What central force law is responsible for this motion?
[**Ans.:** $F(r) = -3\,\dfrac{\ell^2 d}{\mu r^4}$.]

b) Find the total kinetic plus potential energy of this system in the center of mass frame, given that $U(\infty) = 0$.
[Ans.: $E = 0$.]

6. Show that the maximum *radial* velocity of two planets in an elliptical orbit about one another is

$$\dot{r}_{max} = \frac{2\pi\varepsilon a}{\tau(1-\varepsilon^2)^{1/2}}$$

where ε is the eccentricity, τ is the period and "a" is the semimajor axis.

7. Show that the product of the magnitudes of the maximum and minimum *relative* velocities of two bodies in elliptic motion is $(2\pi a/\tau)2$.

8. Starting from (8.110) of the notes, derive to first order in ε only (by a Taylor series expansion) the approximate result,

$$\phi(t) \approx \frac{2\pi t}{\tau} + 2\varepsilon \sin\left(\frac{2\pi t}{\tau}\right).$$

9. (a) Do a simple-minded solution to Kepler's equation, Eq.(8.118) of the notes. Assume ε is small and do an expansion of the form

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_3 + \ldots$$

Find ψ_0, ψ_1 and ψ_2 in terms of the "mean anomoly", $M = \dfrac{2\pi t}{\tau}$.

(b) Show that to order ε you get the same result for $\phi(t)$ as in prob. 8.8 above.

10. Show, for the inverse square law of gravitation ("a" is the major axis, "ε" eccentricity),

$$\text{(a)}\quad \left\langle\frac{1}{r^2}\right\rangle_t = \frac{1}{a^2\sqrt{1-\varepsilon^2}},$$

$$\text{(b)}\quad \left\langle\frac{1}{r^2}\right\rangle_\phi = \frac{1+\frac{\varepsilon^2}{2}}{a^2(1-\varepsilon^2)^2},$$

$$\text{(c)}\quad \left\langle\frac{1}{r^n}\right\rangle_t = \frac{1}{a^2\sqrt{1-\varepsilon^2}}\left\langle\frac{1}{r^{n-2}}\right\rangle_\phi,$$

where

$$\langle\ldots\rangle_t \equiv \frac{1}{\tau}\int_0^\tau \ldots\, dt,$$

$$\langle\ldots\rangle_\phi \equiv \frac{1}{2\pi}\int_0^{2\pi} \ldots\, d\phi.$$

11. (a) Given

$$I_n \equiv \int_0^{2\pi} \frac{d\phi}{(1+\varepsilon\cos\phi)^n},$$

can you find a relation between In and In+1? [Hint: Differentiate in ε.]

(b) Use this to relate $\langle r^n\rangle_t$ to $\langle r^{n-1}\rangle_t$. [See prob. 8.10 for notation.

Ans.: $\langle r^n\rangle_t = \alpha(1-\varepsilon^2)^{3/2}\left(1 + \frac{\varepsilon}{n+1}\frac{d}{d\varepsilon}\right)\left(\frac{\langle r^{n-1}\rangle_t}{(1-\varepsilon^2)^{3/2}}\right)$

12. Verify directly that

$$\langle U(r)\rangle_t = 2E,$$

where $U(r) = -\frac{k}{r}$ and E is the total (internal) energy ($= \frac{\bar{p}^2}{2\mu} - \frac{k}{r}$). You may use any result from prob. 8.10 above to do this.

13. Show that

$$\langle r^n\rangle_t = \frac{1}{a}\langle r^{n+1}\rangle_\psi,$$

where the $\langle\cdots\rangle$ notation indicates an average over an orbital period with respect to the subscripted quantity. ψ is the angular quantity, used in the Kepler equation derivation, given by Eq.(8.113) of the notes:

$$r = a(1 - \varepsilon\cos\psi),$$

where "a" is the semimajor axis and "ε" is eccentricity.

14. Show that

$$\left\langle (\vec{r}\cdot\vec{p})^2 \right\rangle_t = \frac{\mu}{3}\left\langle r^3 \frac{dU}{dr}\right\rangle_t - \frac{\ell^2}{3},$$

($\ell = |\vec{r}\times\vec{p}|$) for any periodic central force motion. You may need a result from the text.

15.(a) Using the idea of effective gravitational potential, show that the turning points in an elliptical orbit under Newton's force law are given by

$$r = \frac{-k \pm \sqrt{k^2 + \dfrac{2E\ell^2}{\mu}}}{2E},$$

where $E < 0$ is the total center of mass energy of the system.

(b) Show that the r_{min}, r_{max} you find agree with Eqs.(8.70), (8.71).

16. A paricle in a nearly circular orbit about a planet suddenly has it's tangential velocity changed by a small amount, δv. Show that

$$2\frac{\delta \ell}{\ell} = \frac{\delta a}{a} = \frac{\delta b}{b} = 2\frac{\delta v}{v} = -\frac{\delta E}{E},$$

where a is the semimajor axis, b is the semiminor axis, v is the magnitude of velocity, E is the (total internal) energy and ℓ is the angular momentum.

Extra credit: The particle's velocity is now changed by an amount δv when it is at r = rmax and ε is no longer small. Show that the changes in the orbital parameters are now:

$$\frac{\delta \ell}{\ell} = \frac{\delta v}{v}, \quad \frac{\delta a}{a} = -\frac{\delta E}{E}, \quad \frac{\delta b}{b} = -\frac{2}{1+\varepsilon}\frac{\delta v}{v}, \quad \frac{\delta E}{E} = -2\frac{1-\varepsilon}{1+\varepsilon}\frac{\delta v}{v}.$$

17. According to the stability analysis, stable circular orbits should exist for a force law of the form,

$$F(r) = -\frac{k}{r^a}, \quad (a<3, k>0).$$

a) Find how the period, τ, is related to the radius of the circular orbit, a (i.e., the analog of Kepler's third law. The text presents two ways of doing this; extra credit if you get the proportionality constant.)
b) Apply the virial theorem to this force law and find how the time averaged kinetic and potentials energies, $\langle T \rangle$ and $\langle U \rangle$, are related to the total energy, E, for circular orbits.

18. A physicist is standing on the Earth's equator when a very strange thing happens (the sort of thing which occurs only on physics homework sets). Suddenly, the Earth's radius shrinks to zero, although it's mass remains unchanged. This leaves the physicist in orbit about the point Earth. Calculate ε (eccentricity) and r_{min} of the physicist's orbit. ($R_E = 6.37\times103$ km, $M_E = 5.98\times10^{27}$ gm.)

Extra credit: Get formulas for ε and rmin when the physicist is located at a latitude λ. What happens when he or she is located at one of the poles?

Other Problems

19. Let us investigate the attractive inverse cube force law, $F(r) = -\frac{k}{r^3}$ $(k>0)$, a bit more.

a) Find the general solutions, r(t) and r(ϕ), when $k = \frac{\ell^2}{\mu}$. (ℓ is the magnitude of the relative angular momentum and μ is the reduced mass.)
b) Describe the qualitative motion of a planet for $\frac{\ell^2}{\mu} > k > 0$. In particular, is there any bound motion?

20. Two point planets (reduced mass μ) are in a circular orbit about one another at a distance r_0 with a force law $F(r) = -\frac{k}{r^a}$ $(1<a<3)$. The orbit is slightly perturbed by a passing comet. Find the period, T, of small oscillations induced by the comet about the circular orbit. Show that in general, this period is not equal to the orbital period, $T_{orbit} = 2\pi\left(\frac{\mu}{k}\right)^{1/2} (r_0)^{(a+1)/2}$.

21. Consider two planets which experience a *repulsive* Coulomb potential, $U(r) = \frac{k}{r}$ $(k>0)$. The total internal energy is E (>0) and the magnitude of the relative angular momentum is ℓ.

a) Show that the distance of closest approach, r_{min}, is given by

$$r_{min} = \frac{k}{2E}\left(1 + \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\right).$$

b) Show that the radius as a function of angle, ϕ, measured from the point of closest approach, is

$$\frac{1}{r} = \frac{k\mu}{\ell^2}\left(-1 + \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\cos(\phi)\right).$$

22. Prob. 8.11(b) resulted in the connection,

$$\langle r^n \rangle_t = \alpha(1-\varepsilon^2)^{3/2}\left(1 + \frac{\varepsilon}{n+1}\frac{d}{d\varepsilon}\right)\left(\frac{\langle r^{n-1}\rangle_t}{(1-\varepsilon^2)^{3/2}}\right),$$

where ε was orbital eccentricity. Using this result twice, show that

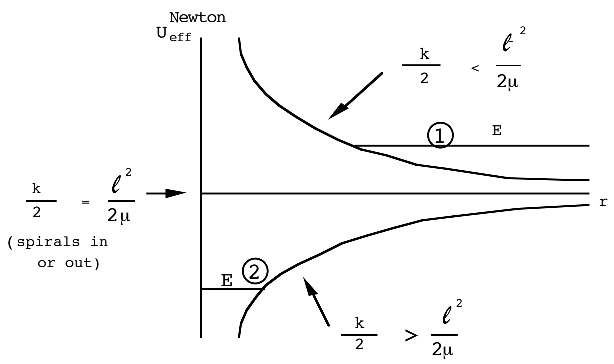
$$\langle r^2 \rangle_t = \alpha^2 \frac{(1+\frac{3}{2}\varepsilon^2)}{(1-\varepsilon^2)^2}.$$

23. The central force responsible for the given trajectory in prob. 8.5 above was $F(r) = -3\frac{\ell^2 d}{\mu r^4}$. I also showed that the total energy of the system, E, was given by E = 0 (given that $U(\infty) = 0$).

a) Draw a graph of the effective potential, $U_{eff}(r)$, vs. r, for this force law. Given E=0 for this trajectory, discuss the types of motion possible.
b) Based upon the $U_{eff}(r)$ graph, and keeping ℓ fixed, find the minimum amount of additional energy, E_{min}, to break free from the force center and move to $r = \infty$.

24. Investigate pure radial motion (ℓ =0) for the $F(r) = -\frac{k}{r^2}$ central force law (k > 0) again, but this time for E > 0 $(U(\infty) = 0)$. Show that a simple parametric representation, similar to prob. 8.4 above (which was for E < 0) is possible. Set the initial conditions to be r=0 at t=0. Solve as completely as possible. Show that your solution approaches the correct velocity, $\frac{p_\infty}{\mu}$, as $t \to \infty$.

25. Given an attractive inverse cubic force law, $F(r) = -\frac{k}{r^3}$ $(k > 0)$, the plot of the effective potential looks like the graph in the text:



Find the orbit solutions, r(ϕ), for cases 1 (total internal energy, E > 0) and 2 (E < 0).

26. Given an attractive force law (k, a > 0)

$$F(r) = -\frac{k}{r^2}e^{-ar},$$

show that circular orbits for r sufficiently small are stable, but for r large they are not. Find the equation which determines the value of r where the transition from stable to unstable orbits occurs.

27. In the text I derived

$$\frac{dA}{dt} = \frac{\ell}{2\mu},$$

where $\ell = |\vec{\ell}|$ $(\vec{\ell} = \vec{r} \times \vec{p})$ and μ is the reduced mass. "A" is the area swept out by the radius vector $\vec{r}$ from the point of view of an observer on m_1 or m_2. Now show that

$$\frac{dA'}{dt} = \mu^2 \left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right) \frac{dA}{dt},$$

where A' is the area swept out by $\vec{r}$ in the center of mass frame of reference. Note this means that

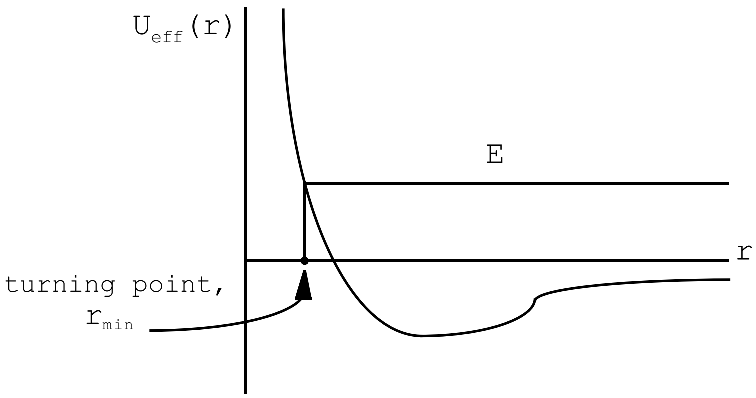
$$\frac{A'}{A} < 1.$$

9 SCATTERING AND COLLISIONS OF PARTICLES

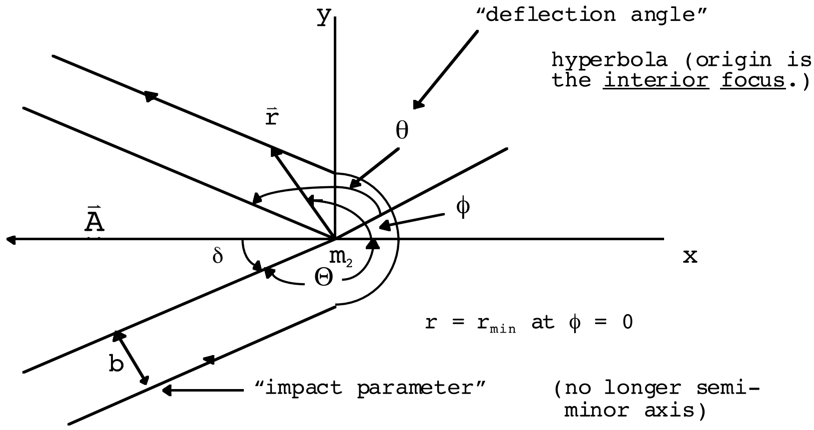
9.1 COULOMB SCATTERING

I will discuss scattering next since it builds on the results of the last chapter.

Consider E > 0 (ε > 1) motion in an attractive Coulomb field. Picture:



Follow the course of $\vec{r}$ during scattering from m_2's point of view:



Equation of orbit (an hyperbola, ε > 1):

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \varepsilon \cos\phi\right), \tag{9.1}$$

$$\frac{1}{r_{min}} = \frac{1}{\alpha}(1 + \varepsilon), \tag{9.2}$$

$$\varepsilon = \left(1 + \frac{2E\ell^2}{\mu k^2}\right)^{1/2}, \tag{9.3}$$

$$\alpha = \frac{\ell^2}{\mu k}. \tag{9.4}$$

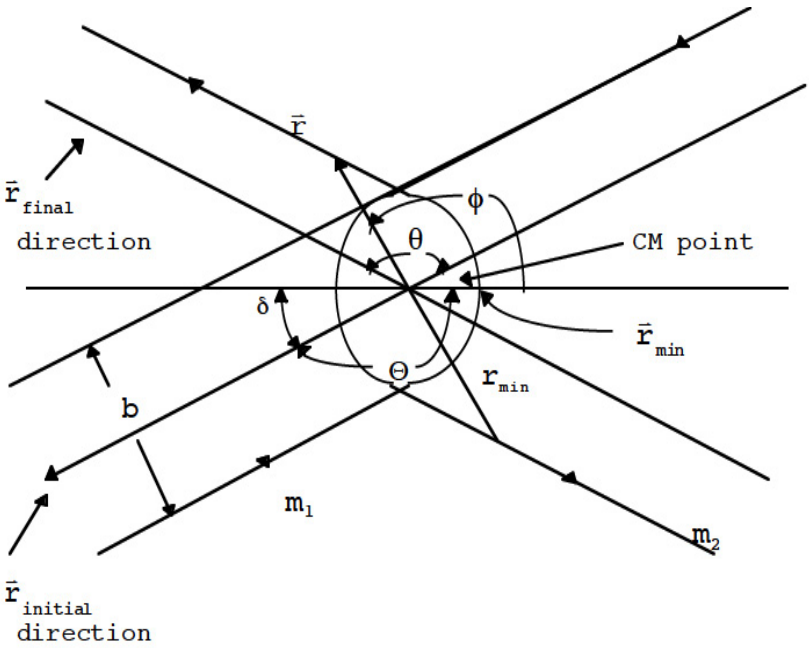
Notice: $2\delta + \theta = \pi$. Also

$$\delta + \Theta = \pi,$$

$$\Rightarrow \Theta = \frac{\pi + \theta}{2}. \tag{9.5}$$

One gets $\Theta = \dfrac{\pi - \theta}{2}$ for the *repulsive* force case. (where the scattering center is now the *exterior* focus of the hyperbola.)

Scattering event looks completely different from CM frame:



Paths are also hyperbolas here ($\vec{r}\,'_1$, $\vec{r}\,'_2$ are just rescaling of $\vec{r}$.) No matter which picture you prefer, it is clear that θ, Θ are quantities relating to the direction of $\vec{r}$ and therefore can be thought of as being measured in any frame. However, we will actually define scattering angles with respect to *velocity* vectors, which will therefore take on different values in alternate inertial frames.

$\theta \equiv$ positive angle between initial and final velocity vectors for either m_1 or m_2 in CM frame.

Things are simpler in the CM; however, this is not the usual experimental situation. Eventually, we will learn how to translate our results in the CM to other frames of reference. Notice that only a certain range of angles for ϕ are now permitted for $\varepsilon > 1$:

$$\frac{1}{r} = \frac{1}{\alpha}(1 + \varepsilon \cos\phi),$$

$$\Rightarrow \frac{1}{\infty} = \frac{1}{\alpha}(1 + \varepsilon \cos\Theta),$$

$$\Rightarrow \cos\Theta = -\frac{1}{\varepsilon} \quad \text{(allows 2 symmetric values of } \Theta\text{; take the + value)} \tag{9.6}$$

$\Rightarrow$ only angles $\cos\phi > -\frac{1}{\varepsilon}$ are allowed. (Values greater than this would say that r is negative.) Go back to the Runge-Lenz vector to see it from another viewpoint:

$$\vec{A} = \vec{\mathcal{L}} \times \vec{p} + \mu k \hat{e}_r,$$

$$\Rightarrow \vec{A} \cdot \vec{p} = \mu k \hat{e}_r \cdot \vec{p}.$$

Remember, $\vec{A}$ is directed along the symmetry Saxis, in the direction opposite to $\vec{r}_{min}$. In particular, if we consider the initial situation with $r \to \infty$,

$$\vec{A} \cdot \vec{p}_\infty = \mu k \hat{e}_{r\infty} \cdot \vec{p}_\infty .$$

$$\vec{A} \cdot \vec{p}_\infty = A p_\infty \cos\Theta \;,\; \hat{e}_{r\infty} \cdot \vec{p}_\infty = -p_\infty ,$$

$$\Rightarrow A p_\infty \cos\Theta = -\mu k p_\infty ,$$

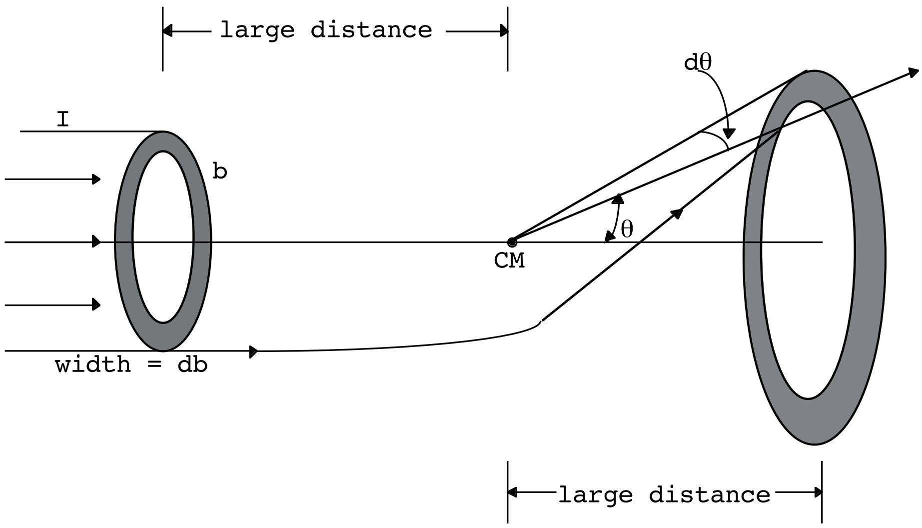
$$\Rightarrow \cos\Theta = -\frac{\mu k}{A} = -\frac{1}{\varepsilon} \text{, as before.}$$

9.2 DIFFERENTIAL CROSS SECTIONS

Need some more concepts for scattering. Let

I = flux in the incident beam (# particles per unit area per unit time)

dN = number of particles through a ring of radius b and width db in the incident beam per unit time.



$$dN = \left(2\pi b \left|db\right|\right) I . \tag{9.7}$$

The particles passing through the ring are scattered through the angles between θ and θ + dθ. dN becomes a scattering concept when we assume that b = b(θ):

$$dN(\theta) = \left(2\pi b(\theta) \left|db(\theta)\right|\right) I . \tag{9.8}$$

$\frac{dN(\theta)}{I}$ = number of particles scattered into (θ, θ + dθ) per unit time per incident flux.

This is now a quantity which is independent of I. We now define the differential cross section as (take $d\theta$ positive)

$$\frac{d\sigma(\theta)}{d\theta} \equiv \frac{dN}{d\theta}\frac{1}{I} = 2\pi\, b(\theta)\left|\frac{db(\theta)}{d\theta}\right|. \tag{9.9}$$

Intrinsically positive. We usually use *solid angle*,

$$d\Omega \equiv \sin\theta\, d\theta d\phi\,. \tag{9.10}$$

We will assume azimuthal symmetry here, so

$$d\Omega = 2\pi \sin\theta d\theta\,,$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\theta) = \frac{b}{\sin\theta}\left|\frac{db(\theta)}{d\theta}\right|. \tag{9.11}$$

9.3 RUTHERFORD SCATTERING IN THE CENTER OF MASS FRAME

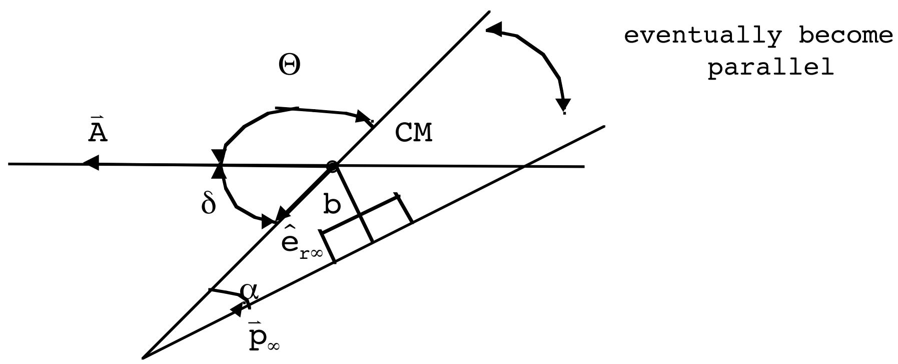
For the Coulomb problem, remember

$$A = \mu^2 k^2 + 2\mu\ell^2 E, \qquad (A = \mu k\varepsilon)$$

$$\Rightarrow 1 = \left(\frac{\mu k}{A}\right)^2 + \frac{2\mu\ell^2 E}{A^2}\,.$$

But $\cos\Theta = -\frac{1}{\varepsilon} = -\frac{\mu k}{A}$,

$$\Rightarrow 1 = \cos^2\Theta + \frac{2\mu\ell^2 E}{A^2}\,.$$



At great distances,

$$E = \text{kinetic} + \text{potential}^{\nearrow 0},$$

$$\Rightarrow E = \frac{p_\infty^2}{2\mu} \qquad \text{(two body problem form)},$$

$$\ell = \left|\bar{r} \times \bar{p}\right| = b\,p_\infty .$$

$$\Rightarrow \sin\Theta = \frac{\ell\sqrt{2\mu E}}{A} = \frac{\ell p_\infty}{A} = \frac{bp_\infty^2}{A},$$

$$\Rightarrow \tan\Theta = -\frac{\frac{bp_\infty^2}{A}}{\frac{\mu k}{A}} = -\frac{bp_\infty^2}{\mu k}.$$

But

$$\Theta = \frac{\theta + \pi}{2} \Rightarrow \tan\Theta = -\cot\frac{\theta}{2},$$

$$\Rightarrow \cot\frac{\theta}{2} = \frac{bp_\infty^2}{\mu k}. \tag{9.12}$$

Provides the necessary connection between b and θ for the Coulomb problem. Now get

$$\frac{db}{d\theta} = \frac{\mu k}{p_\infty^2}\frac{d}{d\theta}\cot\frac{\theta}{2},$$

$$= -\frac{\mu k}{2p_\infty^2}\frac{1}{\sin^2\frac{\theta}{2}}.$$

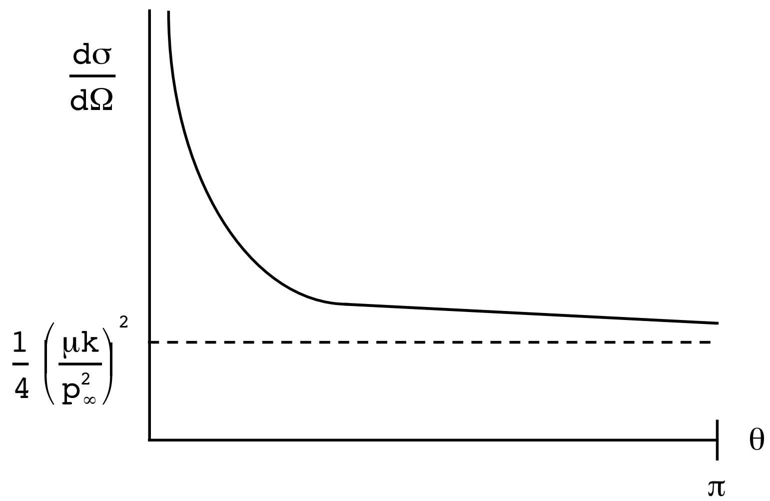
Therefore

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{2}\left(\frac{\mu k}{p_\infty^2}\right)^2\frac{1}{\sin\theta}\cdot\frac{1}{\sin^2\frac{\theta}{2}}\cot\frac{\theta}{2},$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}.\text{“Rutherford formula”} \tag{9.13}$$

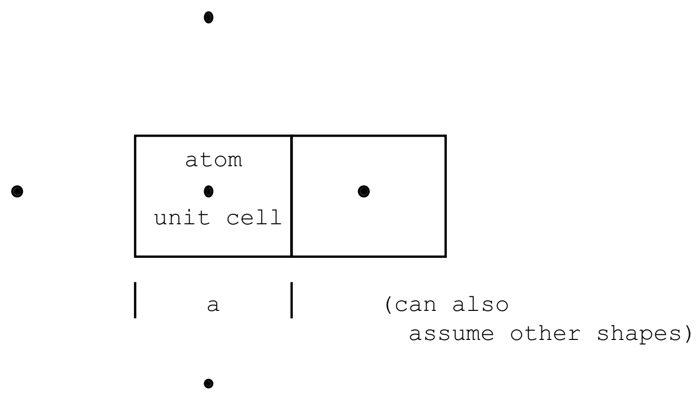
(CM frame)



True also for *repulsive* Coulomb force. Indeed, unchanged by quantum mechanics (non-relativistic), except that $k \to \pm(Zze^2)$ for electrically charged particles. (Interpretation of it in quantum mechanics is completely different, however.)

First measured experimentally by H. Geiger and E. Marsden. By careful observation in a darkened room, they found (by counting!) that approximately one in every eight thousand α-particles (Helium nuclei) was backscattered from a thin gold target. Let's see if we can understand the $\frac{1}{8000}$ factor backscattering fraction from the above formula.

Model: (each atomic layer)



Assume layers are randomly oriented with respect to each other (i.e., one atom in front does not "shadow" an atom behind.)

Each α-particle can be imagined to pass through the unit cell shown as may times as there are layers of atoms. We will take "backscattering" to mean $\frac{\pi}{2} < \theta < \pi$.

$$\frac{d\sigma}{d\Omega} = \frac{dN}{d\Omega}\frac{1}{I} = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}},$$

$$dN = I\,d\sigma = \frac{I}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{d\Omega}{\sin^4\frac{\theta}{2}},$$

$$N_{back} = I\sigma_{back} = \frac{1}{4} I \left(\frac{\mu k}{p_\infty^2}\right)^2 \int_{\pi/2}^{\pi} \frac{d\Omega}{\sin^4\frac{\theta}{2}}.$$

For this rough estimate, we will set $\sin^4\frac{\theta}{2} \to 1$ over this range of θ, and since the solid angle corresponding to $\frac{\pi}{2} < \theta < \pi$ is 2π is 2π, we then have approximately,

$$N_{back} \simeq \frac{\pi}{2} I \left(\frac{\mu k}{p_\infty^2}\right)^2$$

For scattering α's off gold foil:

$$k \to Zze^2,\ \mu \simeq m_\alpha,$$

$$Z = 79\ ,\ z = 2,$$

$$\Rightarrow N_{back} \simeq \frac{\pi}{2} I \left(\frac{Zze^2 m_\alpha}{m_\alpha^2 v_\alpha^2}\right)^2.$$

Let us assume that (nonrelativistic formula)

$$\frac{1}{2} m_\alpha v_\alpha^2 \simeq 5\text{MeV} \qquad \text{(I looked it up)}$$

$(1 \text{ MeV} \simeq 1.6 \times 10^{-6} \text{erg})$. We also need

$$|e| = 4.803 \quad 10^{-10}\text{esu}$$
$$m_\alpha = 6.68 \quad 10^{-24}\text{gm}$$

$$\Rightarrow p_\infty = m_\alpha v_\alpha = 1.034 \times 10^{-14}\text{gm}\frac{\text{cm}}{\text{sec}}.$$

Still need I. In our case (imagine a single particle passing through the sample; also imagine multiplying both sides of the above equation for N_{back} by the total time of the experiment so that N_{back} is a pure number):

I = number scattering processes per alpha particle / unit cell

$$\Rightarrow I = \frac{1}{a^2} n,$$

where n is the number of atomic layers in the sample thickness. Sample thickness was ~ 1μ = 10^{-4}cm (also looked this up). So

$$n = \frac{10^{-4}\,\text{cm}}{a},$$

$$\Rightarrow I = \frac{10^{-4}\,\text{cm}}{a^3}$$

Some other calculations giving a:

$$\rho_{\text{gold}} = 19.28\,\frac{\text{gm}}{\text{cm}^3}$$

$$N_A = \overset{\substack{\underline{\text{Avagadro}} \\ \downarrow}}{6.023} \times 10^{23}\,\frac{\text{amu}}{\text{gm}}, \quad m_{\text{gold}} \simeq 197\,\text{amu}$$

$$\left(\frac{\text{atoms}}{\text{cm}^3}\right)_{\text{gold}} = \rho_{\text{gold}} \times \left(\frac{\#\,\text{atoms}}{\text{gm}}\right) = 19.28\left(\frac{6.023 \times 10^{23}}{197}\right)$$

$$= 5.9 \times 10^{22}\,\frac{\text{atoms}}{\text{cm}^3}.$$

This means

$$a \approx \left(5.9 \times 10^{22}\right)^{-1/3} = 2.57 \times 10^{-8}\,\text{cm}$$

$$\Rightarrow I \approx 5.90 \times 10^{18}\,\text{cm}^{-2}.$$

Putting all the pieces together now gives us

$$\Rightarrow N_{\text{back}} \approx 4.76 \times 10^{-5} \approx \frac{1}{21{,}000}.$$

too small ~ 2. One reason: did not integrate the cross section. This means we have underestimated the number of interactions. (But see the homework problem!)

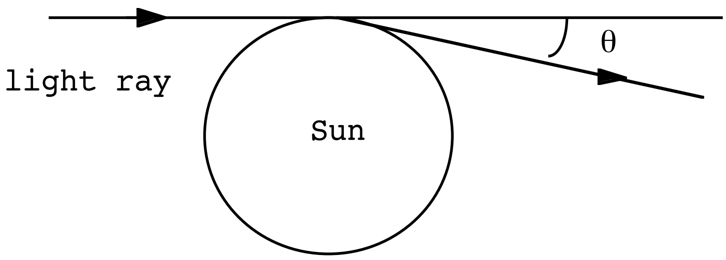
9.4 SIMPLE TREATMENT OF LIGHT DEFLECTION

Since we talked a little about one general relativity effect last chapter (perihelion procession), will indicate another result here. For very small angular defections, the above relationship between b and θ becomes

$$b \approx \frac{\mu k}{p_\infty^2} \frac{1}{\theta/2}$$

$$\Rightarrow \theta \approx \frac{2\mu k}{p_\infty^2} \frac{1}{b} \qquad \left(\mu = \frac{m_1 m_2}{m_1 + m_2}, k = Gm_1 m_2\right)$$

Use this to model a light ray grazing the radius of the Sun:



Therefore, take $b = R_S$, $m_2 = M_S$ (Sun's radius, mass.) What to do about $\mu \simeq m_1$, p_∞? For light,

$$p = \frac{h\nu}{c}.$$

Use the relativistic connection between mass and energy to replace (E = pc)

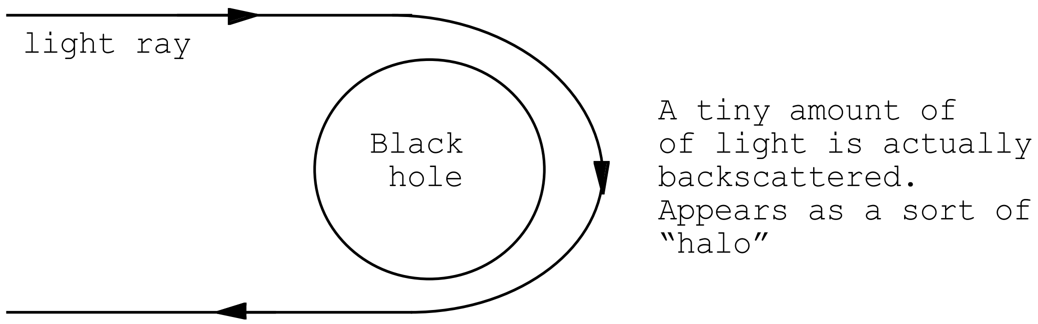
$$m_1 \rightarrow \frac{E}{c^2} = \frac{h\nu}{c^2},$$

$$\Rightarrow \theta \approx \frac{2\frac{(h\nu)}{c^2} GM_S \frac{(h\nu)}{c^2}}{\left(\frac{h\nu}{c}\right)^2 R_S} = \frac{2GM_S}{c^2 R_S}.$$

Almost correct, but too small by a factor of 2! (This was, in fact, Einstein's original result, which he modified later.) Einstein's general relativity gives the correct result:

$$\theta \approx \frac{4GM_S}{c^2 R_S} = 1.75\text{" (sec) of arc}.$$

Another new thing about light scattering in general relativity: *glories*.



9.5 CROSS SECTION COOKBOOK

We have been discussing a special case of scattering, the inverse square force law. We need some cookbook formulas for doing other force laws. Will give differential cross section in the CM frame. Need general connection between b and θ. Start:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)\,.$$

Can "reduce the problem to quadratures." Solve for $\dot{r}$:

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{\ell^2}{m^2 r^2}}\,, \tag{9.14}$$

↑
have to pick correct root
($U_{eff}(r)$ picture good for this)

$$d\phi = \frac{d\phi}{dt}\frac{dt}{dr}dr = \frac{\dot{\phi}}{\dot{r}}dr\,,\ \text{but}\ \dot{\phi} = \frac{\ell}{\mu r^2}\,,$$

$$d\phi = \pm\frac{\ell/r^2\ dr}{\sqrt{2\mu\left(E - U(r)\right) - \frac{\ell^2}{r^2}}} \tag{9.15}$$

Assuming $E > 0$, $\left(E = \frac{p_\infty^2}{2\mu}\right)$ and *integrating* on dr from r_{min} to $r = \infty$, we get

$r = \infty$, we get ($\ell = b p_\infty$):

Θ defined positive

↓

$$\Theta = +b\int_{r_{min}}^{\infty}\frac{dr/r}{\sqrt{\left(r^2 - b^2\right) - \frac{2\mu}{p_\infty^2}r^2 U(r)}}\,. \tag{9.16}$$

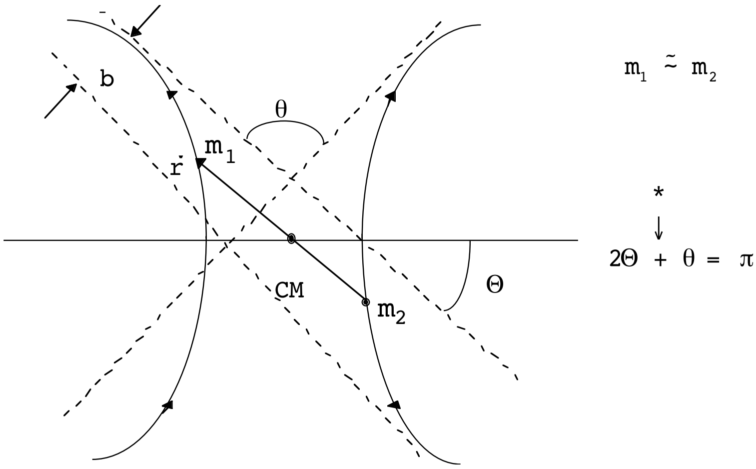
Warning: r_{min} itself is a function of b, in general.

Cookbook steps:

1. Evaluate $r_{min}(b)$.
2. Do integral.
3. Use $\Theta = \frac{\pi + \theta}{2}$ or $\Theta = \frac{\pi - \theta}{2}$ in the attractive or repulsive cases, respectively, to find $b(\theta)$.
4. Plug in $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$.

Another warning: does not work for all potentials, U(r). Most require $\lim_{r\to\infty} U(r) = 0$.

Just to get a feeling for using the cookbook method, do the *repulsive* Coulomb case (different from attractive case). Picture:



Jump to result: (you will verify this in a problem)

$$\cos\Theta = \frac{\dfrac{k\mu}{p_\infty^2 b}}{\sqrt{1+\left(\dfrac{k\mu}{p_\infty^2 b}\right)^2}} \tag{9.17}$$

Can write as

$$\cos^2\Theta = \frac{\left(\dfrac{k\mu}{p_\infty^2 b}\right)^2}{1+\left(\dfrac{k\mu}{p_\infty^2 b}\right)^2},$$

$$\Rightarrow \tan^2\Theta = \left(\frac{p_\infty^2 b}{k\mu}\right)^2 \text{ or } b^2 = \left(\frac{k\mu}{p_\infty^2}\right)^2 \tan^2\Theta$$

Choose $b = +\left(\dfrac{k\mu}{p_\infty^2}\right)\tan\Theta$, $\Theta = \dfrac{\pi-\theta}{2}$,

↑

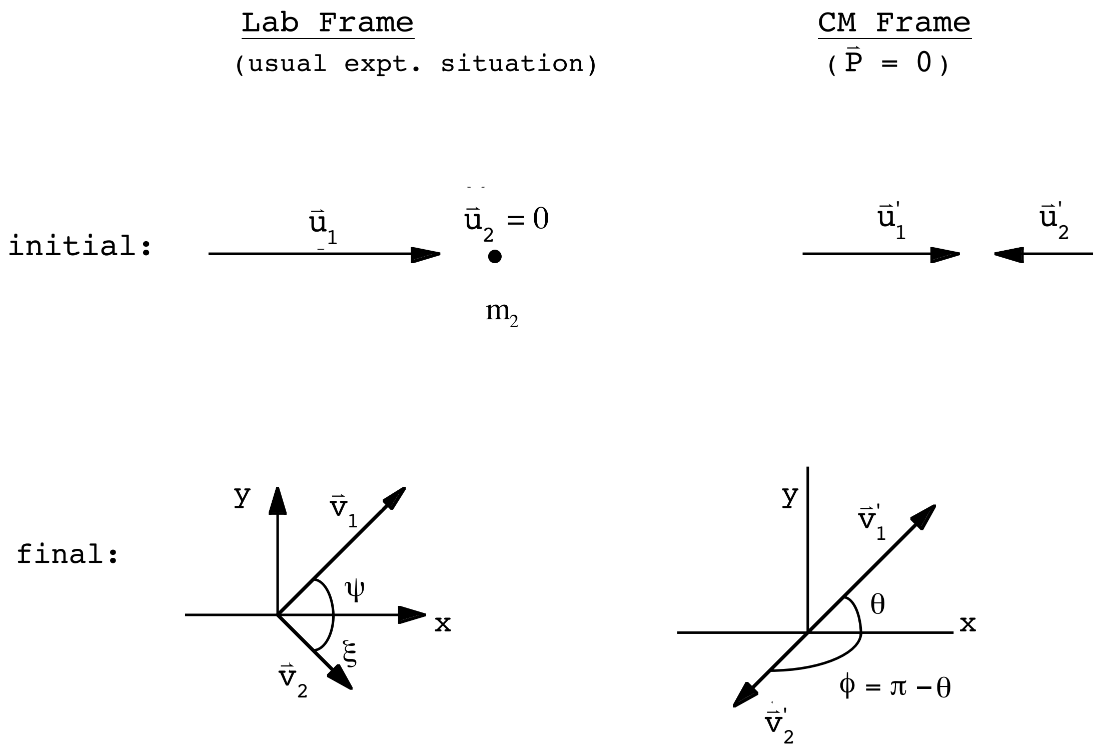
different from attractive case

$$\Rightarrow b = \frac{k\mu}{p_\infty^2}\cot\frac{\theta}{2},$$

same as before.

9.6 CONNECTION BETWEEN LAB AND CM FRAMES

Only problem: cross sections usually not measured in the CM frame.



Only showing initial and final velocities; not discussing dynamics but kniematics. We will assume *elastic* collisions – no heat generated or mass/energy lost. We will conserve both momentum and energy. We will also assume that all motion takes place in one plane (azimuthal symmetry, as we assumed before in the cross section discussion). We will take the angles ψ, ξ, etc. as positive; if not, we can always re-orient our axis so that they are.

Only difference between the two viewpoints: observed by two people who have a relative velocity, $\vec{V}$. All primed and umprimed quantities are related by $\vec{V}$:

initial		final	
$\vec{u}_1 = \vec{u}_1' + \vec{V}$	(9.18a)	$\vec{v}_1 = \vec{v}_1' + \vec{V}$	(9.19a)
$\vec{u}_2 = \vec{u}_2' + \vec{V} = 0$	(9.18b)	$\vec{v}_2 = \vec{v}_2' + \vec{V}$	(9.19b)

$$\Rightarrow \vec{u}_2' = -\vec{V}$$

notation

$\vec{u}$ - initial velocity

$\vec{v}$ - final velocity

1,2 - which particle

prime, unprimed - CM, Lab frame, respectively

By definition,

$$\vec{R} = \frac{1}{M}\sum_i m_i \underset{\text{lab coordinates}}{\vec{r}_i},$$

$$\vec{R} = \frac{1}{m_1 + m_2}\left(m_1\vec{r}_1 + m_2\vec{r}_2\right),$$

$$\Rightarrow \vec{V} = \frac{1}{m_1 + m_2}\left(m_1\vec{u}_1 + m_2\vec{u}_2\right), \quad \left(\vec{u}_2 = 0\right)$$

$$\vec{V} = \frac{m_1\vec{u}_1}{m_1 + m_2}. \qquad \left(= -\vec{u}_2'\right) \tag{9.20}$$

In the CM coordinate system $\left(\vec{P}\right)_{\text{before}}^{\text{after}} = 0$. Therefore

$$m_1\vec{u}_1' + m_2\vec{u}_2' = 0 \Rightarrow m_1u_1' = m_2u_2', \tag{9.21}$$

$$m_1\vec{v}_1' + m_2\vec{v}_2' = 0 \Rightarrow m_1v_1' = m_2v_2' \tag{9.22}$$

(u_1', v_1' etc. are magnitudes only.) Assume any potential that exists between the particles ∅ 0 as distances ∅ ∞; then, far enough apart, the energy is purely kinetic. Let

T_0' = total energy in CM frame

$$\left(T_0'\right)_{\text{before}} = \left(T_0'\right)_{\text{after}},$$

$$\frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2,$$

$$\Rightarrow m_1u_1'^2 + \left(\frac{m_1}{m_2}\right)^2 m_2u_1'^2 = m_1v_1'^2 + m_2\left(\frac{m_1}{m_2}\right)^2 v_1'^2,$$

$$\Rightarrow u_1' = v_1'. \tag{9.23}$$

Also

$$m_1\left(\frac{m_1}{m_2}\right)^2 u_2'^2 + m_2u_2'^2 = \left(\frac{m_1}{m_2}\right)^2 m_1v_2'^2 + m_2v_2'^2,$$

$$\Rightarrow u_2' = v_2'. \tag{9.24}$$

Thus, to summarize:

$$u_1' = v_1' = \frac{m_2}{m_1}u_2' = \frac{m_2}{m_1}v_2'.$$

⇒ only 1 unknown velocity magnitude in the CM frame. Other unknown: θ (ϕ = π – θ). Let's say we measure these 2 things in a given collision. How are they related to quantities in the Lab frame? From before,

$$\vec{V} = \frac{m_1\vec{u}_1}{m_1 + m_2} = -\vec{u}_2',$$

but

$$u_2' = \frac{m_1}{m_2} u_1',$$

$$\Rightarrow u_1 = \frac{m_1 + m_2}{m_1} \cdot \frac{m_1}{m_2} u_1' = \left(1 + \frac{m_1}{m_2}\right) \underset{\uparrow}{\underline{\underline{u_1'}}} \tag{9.25}$$

known or measured, say

Not specified yet: $\vec{v}_1$, $\vec{v}_2$. We have,

$$\vec{v}_1 = \vec{v}_1' + \vec{V},$$

$$\vec{v}_2 = \vec{v}_2' + \vec{V}.$$

Have to start invoking angles now:

$$\begin{aligned} v_1^2 &= v_1'^2 + V^2 + 2\vec{V} \cdot \vec{v}_1', \\ v_2^2 &= v_2'^2 + V^2 + 2\vec{V} \cdot \vec{v}_2'. \end{aligned} \left(\begin{aligned} \vec{V} &= -\vec{u}_2' \\ \Rightarrow V &= u_2' = \frac{m_1}{m_2} u_1' \end{aligned} \right)$$

$$\begin{array}{l} \text{refer to} \\ \text{the figures} \end{array} \rightarrow \left\{ \begin{aligned} 2\vec{V} \cdot \vec{v}_1' &= 2Vv_1' \cos\theta = 2\frac{m_1}{m_2} u_1' u_1' \cos\theta \\ 2\vec{V} \cdot \vec{v}_2' &= 2Vv_2' \cos(\pi - \theta) = -2Vv_2' \cos\theta \end{aligned} \right.$$

$$= -2\frac{m_1}{m_2} u_1' \frac{m_1}{m_2} u_1' \cos\theta$$

$$v_1^2 = u_1'^2 + \left(\frac{m_1}{m_2}\right)^2 u_1'^2 + 2\frac{m_1}{m_2} u_1'^2 \cos\theta,$$

$$\Rightarrow \quad v_1^2 = u_1'^2 \left[1 + \left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right)\cos\theta\right] \tag{9.26}$$

$$v_2^2 = \left(\frac{m_1}{m_2}u_1'\right)^2 + \left(\frac{m_1}{m_2}u_1'\right)^2 - 2\left(\frac{m_1}{m_2}u_1'\right)^2 \cos\theta,$$

$$v_2^2 = 2\left(\frac{m_1}{m_2}u_1'\right)^2 (1 - \cos\theta),$$

$$v_2^2 = 4u_1'^2\left(\frac{m_1}{m_2}\right)^2 \sin^2\frac{\theta}{2}. \tag{9.27}$$

Therefore v_1, v_2 are known if u_1', θ are known. Only things left: ψ, ξ

① $\vec{v}_1 = \vec{v}_1' + \vec{V}$,

② $\vec{v}_2 = \vec{v}_2' + \vec{V}$.

① $$x \;:\; v_1 \cos\psi = v_1' \cos\theta + V, \tag{9.28}$$

$$y \;:\; v_1 \sin\psi = v_1' \sin\theta. \tag{9.29}$$

$$\text{2} \quad x : \quad v_2 \cos\xi = v_2' \underbrace{\cos(\pi-\theta)}_{-\cos\theta} + V, \tag{9.30}$$

$$y : \quad -v_2 \sin\xi = -v_2' \underbrace{\sin(\pi-\theta)}_{\sin\theta}. \tag{9.31}$$

Divide $\frac{y}{x}$ from 1:

$$\tan\psi = \frac{v_1' \sin\theta}{v_1' \cos\theta + V} = \frac{\sin\theta}{\cos\theta + \frac{V}{v_1'}}. \tag{9.32}$$

$$\text{But } \frac{V}{v_1'} = \frac{\frac{m_1}{m} u_1'}{u_1'} = \frac{m_1}{m_2} \quad\Rightarrow\quad \tan\psi = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}. \tag{9.33}$$

Says that the connection between θ and ψ is *not unique* under some circumstances. Look at $\frac{m_1}{m_2} >> 1$:

$$\tan\psi \approx \frac{m_2}{m_1} \sin\theta \tag{9.34}$$

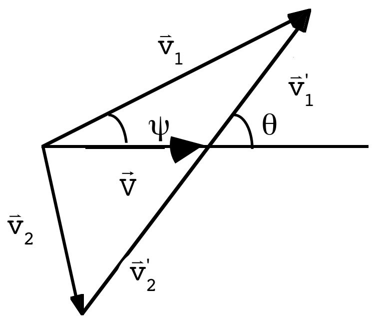
Given ψ, 2 values for θ. Other extreme, $\frac{m_1}{m_2} << 1$:

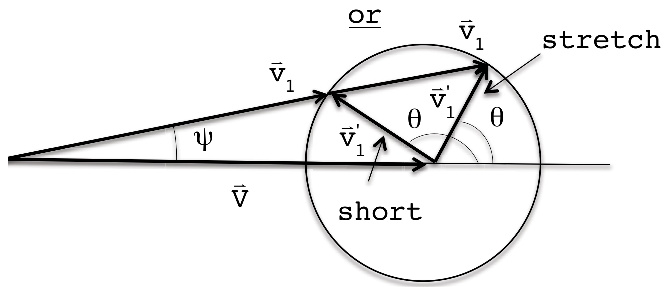
$$\tan\psi \approx \tan\theta,$$

$$\Rightarrow \quad \psi \approx \theta. \tag{9.35}$$

One value of $\psi \Rightarrow$ one value of θ. Geometry helps:

$$\frac{V}{v_1'} = \frac{m_1}{m_2} < 1 \qquad\qquad \frac{V}{v_1'} = \frac{m_1}{m_2} > 1$$



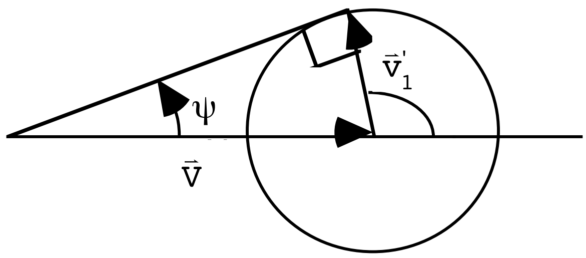


one value of $\psi \Rightarrow$
one value of θ

one value of $\psi \Rightarrow$
two values of θ

However, ψ *and* v_1 determine θ uniquely. Also, one value of $\theta \Rightarrow$ always one value of ψ. Can now see what the two extreme cases above correspond to geometrically.

More geometry: $\dfrac{m_1}{m_2} > 1$ case:



There is a max value of ψ in this case:

$$\sin\,\psi_{max} = \frac{v_1'}{V} = \frac{m_2}{m_1}\,. \tag{9.36}$$

In this case one value of $\psi \Rightarrow$ one value of θ. Rutherford case: $\dfrac{m_1}{m_2} << 1$, essentially scattering individual α-particles off the entire sample since the atomic centers are fixed => unique determination.

Special case: $m_1 = m_2$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + 1} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}\,,$$

$$" \quad = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2},$$

$$\Rightarrow \psi = \frac{\theta}{2}. \tag{9.37}$$

Now consider equation (2) above. Divide $\frac{y}{x}$:

$$\frac{v_2 \sin\xi}{v_2 \cos\xi} = \frac{v_2' \sin\theta}{-v_2' \cos\theta + V},$$

$$\Rightarrow \tan\xi = \frac{\sin\theta}{-\cos\theta + \frac{V}{v_2'}}.$$

However, $V = v_2'$,

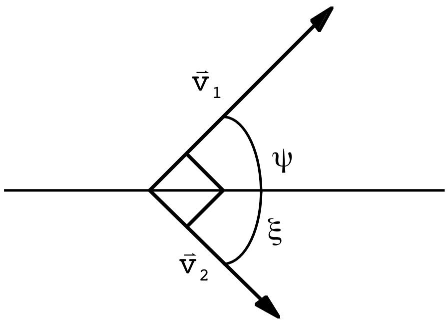
$$\tan\xi = \frac{\sin\theta}{-\cos\theta + 1} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}},$$

$$\Rightarrow \tan\xi = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right),$$

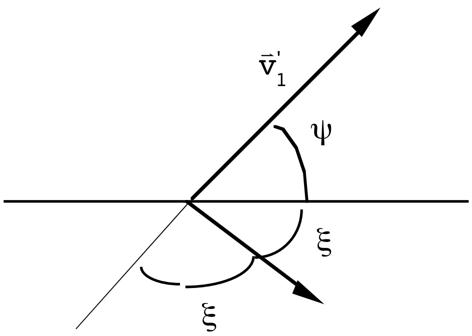
$$\Rightarrow \xi = \frac{\pi - \theta}{2} \overset{\phi=\pi-\theta}{\Rightarrow} \xi = \frac{\phi}{2}. \tag{9.38}$$

For the $m_1 = m_2$ case we had $\psi = \dfrac{\theta}{2}$, so

$$\xi + \psi = \frac{\pi}{2} \quad , \quad (m_1 = m_2) \tag{9.39}$$



In the $\dfrac{m_1}{m_2} << 1$ case:



$$2\xi = \pi - \theta,$$

$$\theta \approx \psi,$$

$$\Rightarrow 2\xi + \psi \approx \pi. \tag{9.40}$$

Let's find relationships between the various kinetic energies. Definitions first:

$$\begin{pmatrix} T_0 \\ T_0' \end{pmatrix} = \text{total K.E. in} \begin{pmatrix} \text{lab} \\ \text{CM} \end{pmatrix} \text{frame.}$$

$$\begin{pmatrix} T_1 \\ T_1' \end{pmatrix} = \text{final K.E. of } m_1 \text{ in} \begin{pmatrix} \text{lab} \\ \text{CM} \end{pmatrix} \text{frame.}$$

Similarly for T_2, T_2'. Of course

$$T_0' = T_1' + T_2' \quad , \quad T_0 = T_1 + T_2.$$

Now

$$T_0 = \frac{1}{2} m_1 u_1^2, \tag{9.41}$$

$$T_0' = \frac{1}{2}\left(m_1 u_1'^2 + m_2 u_2'^2\right) \tag{9.42}$$

$$\begin{pmatrix} T_1' = \frac{1}{2} m_1 v_1'^2 \\ T_1 = \frac{1}{2} m_1 v_1^2 \end{pmatrix} \qquad \begin{pmatrix} T_2' = \frac{1}{2} m_2 v_2'^2 \\ T_2 = \frac{1}{2} m_2 v_2^2 \end{pmatrix}$$

Let's try to express them all in terms of T_0. Remember

$$u_1 = u_1'\left(1 + \frac{m_1}{m_2}\right) \quad , \quad u_1' = \frac{m_2}{m_1} u_2',$$

$$\Rightarrow \quad T_0' = \frac{1}{2}\left(\frac{m_1}{\left(1 + \frac{m_1}{m_2}\right)^2} + \frac{m_2\left(\frac{m_1}{m_2}\right)^2}{\left(1 + \frac{m_1}{m_2}\right)^2}\right) u_1^2,$$

$$\Rightarrow \quad T_0' = \frac{1}{2}\left[\frac{m_1 m_2^2 + m_2 m_1^2}{\left(m_1 + m_2\right)^2}\right] u_1^2 = \frac{1}{2}\underset{\substack{\uparrow \\ \text{reduced mass}}}{\mu} u_1^2.$$

$$\Rightarrow T_0' = \frac{\mu}{m_1} T_0. \tag{9.43}$$

so $T_0' < T_0$ always. Also

$$T_1' = \frac{1}{2} m_1 v_1'^2 = \frac{1}{2} m_1 u_1'^2 = \frac{\frac{1}{2} m_1 u_1^2}{\left(1 + \frac{m_1}{m_2}\right)^2},$$

$$\Rightarrow T_1' = \left(\frac{m_2}{m_1 + m_2}\right)^2 T_0. \tag{9.44}$$

$$T_2' = \frac{1}{2} m_2 v_2'^2 = \frac{1}{2} m_2 \left(\frac{m_1}{m_2} u_1' \right)^2 = \frac{\frac{1}{2} m_2 \left(\frac{m_1}{m_2} \right)^2 u_1^2}{\left(1 + \frac{m_1}{m_2} \right)^2} ,$$

$$\Rightarrow T_2' = \frac{m_1 m_2}{\left(m_1 + m_2 \right)^2} T_0 . \tag{9.45}$$

We have $\frac{T_1}{T_0} = \frac{v_1^2}{u_1^2}$. Go back to earlier v_1 expression:

$$v_1 = u_1' \sqrt{1 + \left(\frac{m_1}{m_2} \right)^2 + 2 \left(\frac{m_1}{m_2} \right) \cos\theta} ,$$

$$\Rightarrow \quad v_1 = \frac{u_1}{\left(1 + \left(\frac{m_1}{m_2} \right) \right)} \sqrt{1 + \left(\frac{m_1}{m_2} \right)^2 + 2 \left(\frac{m_1}{m_2} \right) \cos\theta} ,$$

$$\Rightarrow \quad \left(\frac{v_1}{u_1}\right)^2 = \left(\frac{m_2^2 + m_1^2 + 2m_1 m_2 \cos\theta}{(m_1 + m_2)^2}\right) = \frac{T_1}{T_0}\,. \tag{9.46}$$

What about in terms of ψ? Must express $\cos\theta$ as a function of ψ.

$$\tan\psi = \frac{\sin\theta}{\cos\theta + x}\,. \qquad \left(x \equiv \frac{m_1}{m_2}\right) \tag{9.47}$$

[“Aside 1”:

$$\Rightarrow x = \frac{\sin\theta}{\tan\psi} - \cos\theta\,,$$

$$x = \frac{\sin\theta\cos\psi - \cos\theta\sin\psi}{\sin\psi}\,,$$

$$x = \frac{\sin(\theta - \psi)}{\sin\psi} \quad , \quad \text{useful later.}\ ]$$

$$\Rightarrow \quad \tan^2\psi = \frac{1 - \cos^2\theta}{\cos^2\theta + x^2 + 2x\cos\theta}\,,$$

$$\tan^2\psi\left(\cos^2\theta + x^2 + 2x\cos\theta\right) = 1 - \cos^2\theta\,,$$

$$\cos^2\theta\,\underbrace{\left(\tan^2\psi + 1\right)}_{\uparrow\ \frac{1}{\cos^2\psi}} + \cos\theta\left(2x\tan^2\psi\right) + \left(x^2\tan^2\psi - 1\right) = 0\,.$$

A quadratic equation in $\cos\theta$. Solution to $Ax^2 + Bx + C = 0$ is

$$x = \frac{-B \pm \sqrt{B^2 - 4ac}}{2A}\,,$$

so

$$\cos\theta = \frac{-2x\tan^2\psi \pm \sqrt{4x^2\tan^4\psi + \dfrac{4\left(1 - x^2\tan^2\psi\right)}{\cos^2\psi}}}{\dfrac{2}{\cos^2\psi}},$$

$$\Rightarrow \cos\theta = -x\sin^2\psi \pm \cos^2\psi\sqrt{x^2\tan^4\psi + \frac{\left(1 - x^2\tan^2\psi\right)}{\cos^2\psi}}. \tag{9.48}$$

Inside the square root:

$$x^2\tan^2\psi\underbrace{\left(\tan^2\psi - \frac{1}{\cos^2\psi}\right)}_{-1} = -x^2\tan^2\psi,$$

$$\Rightarrow \cos\theta = -x\sin^2\psi \pm \cos^2\psi\sqrt{\frac{1}{\cos^2\psi} - x^2\tan^2\psi}, \tag{9.49}$$

$\uparrow$

would have written $|\cos\psi|$, but because of the $\pm$ signs, this does not matter.

From above

$$\frac{T_1}{T_0} = \frac{\left[1 + \dfrac{1}{x^2} + \dfrac{2}{x}\cos\theta\right]}{\left(1 + \dfrac{1}{x}\right)^2}, \tag{9.50}$$

so

$$\frac{T_1}{T_0} = \frac{\left[1 + \dfrac{1}{x^2} - 2\sin^2\psi \pm 2\cos\psi\sqrt{\dfrac{1}{x^2} - \sin^2\psi}\right]}{\left(1 + \dfrac{1}{x}\right)^2}. \tag{9.51}$$

Notice the numerator is a perfect square:

$$\left(\cos\psi \pm \sqrt{\frac{1}{x^2} - \sin^2\psi}\right)^2 = \cos^2\psi + \frac{1}{x^2} - \sin^2\psi$$

$$\pm 2\cos\psi\sqrt{\frac{1}{x^2} - \sin^2\psi}\,,$$

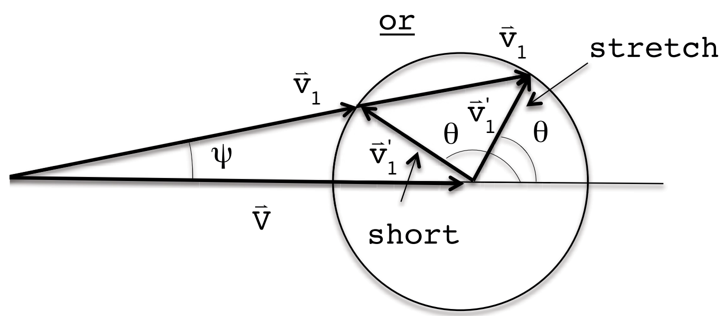
$$= 1 + \frac{1}{x^2} - 2\sin^2\psi \pm 2\cos\psi\sqrt{\frac{1}{x^2} - \sin^2\psi}\,.$$

$$\Rightarrow \frac{T_1}{T_0} = \frac{1}{\left(1 + \frac{1}{x}\right)^2}\left[\cos\psi \pm \sqrt{\frac{1}{x^2} - \sin^2\psi}\right]^2 \tag{9.52}$$

Since $\dfrac{T_1}{T_0} = \left(\dfrac{v_1}{u_1}\right)^2$,

$$\Rightarrow v_1 = \frac{u_1}{(1 + x)}\left[x\cos\psi \pm \sqrt{1 - x^2\sin^2\psi}\right]. \tag{9.53}$$

The ± corresponds to the 2 possibilites in the following figure. Remember, for $\frac{m_1}{m_2} = x > 1$, $\psi_{max} < \frac{\pi}{2} \Rightarrow \cos\psi > 0$ So we have



$$v_1 = \frac{u_1}{(1+x)}\left[x\cos\psi \underset{\uparrow\ \text{``short''}\ (x>1)}{\overset{\text{“stretch”}\ \downarrow}{\pm}} \sqrt{1 - x^2\sin^2\psi}\right]$$

Confirmation: “short” case v_1 should $\rightarrow 0$ as $x \rightarrow 1^+$ (through larger values of x):

$$v_{1\,\text{short}} \rightarrow \frac{u_1}{2}\underbrace{\left[\cos\psi - \sqrt{1 - \sin^2\psi}\right]}_{\cos\psi - |\cos\psi|} = 0\,.$$

⇒ Only the “stretch” case survives for x < 1. (Can see that for x < 1 the negative root would make v_1 negative.) The same interpretation applies to the expression for $\frac{T_1}{T_0}$.

[“Aside 2”: From 1, y equation:

$$v_1 \sin\psi = v_1' \sin\theta$$

[“Aside 2”: From 1, y equation:

$$v_1 \sin\psi = v_1' \sin\theta$$

$$\Rightarrow \frac{\sin\theta}{\sin\psi} = \frac{v_1}{v_1'} = \frac{\frac{u_1}{(1+x)}\left[x\cos\psi \pm \sqrt{1 - x^2\sin^2\psi}\right]}{\frac{u_1}{(1+x)}},$$

"stretch"
↓
$$\cos(\theta - \psi) = \pm\sqrt{1 - x^2 \sin^2 \psi}\,.\ ]$$
↑
''short''

["Aside 3": From "Aside 1" we have

$$\sin(\theta - \psi) = x \sin \psi\,.$$

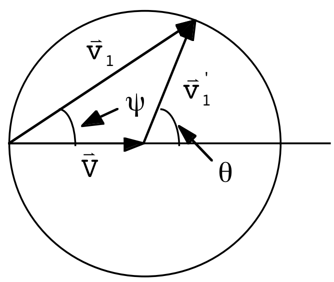
Thus, from

$$\sin^2(\theta - \psi) + \cos^2(\theta - \psi) = 1\,,$$

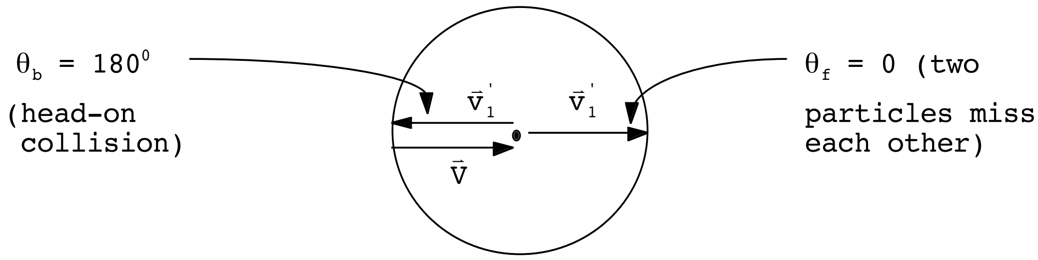
we have immediately

"stretch"
↓
$$\cos(\theta - \psi) = \pm\sqrt{1 - x^2 \sin^2 \psi}\,.\ ]$$
↑
''short''

We get very simple results when $m_1 = m_2$. Then, for $\psi \neq 0$, we have only one θ value:



There are actually two solutions where $\vec{V}$ and $\vec{v}_1^{\,'}$ are co-linear in this case:



The solution where the particles miss each other has $\psi = 0$ whereas the head-on collision is characterized by $\psi \rightarrow 90^0$. (Remember, $\psi = \frac{\theta}{2}$ for $x = 1$. We have for $x = 1$,

$$\frac{T_1}{T_0} = \frac{1}{(1+1)^2}\left[2\cos\psi\right]^2 = \cos^2\psi\,. \tag{9.54}$$

This goes to zero as $\psi \rightarrow 90^0$, which means in this limit m_1 comes to a complete halt after the collision. This fact is useful in nuclear reactor modulators, which slow down (or moderate) neutrons. It says that the best way of slowing down free neutrons is a material which contains light nuclei (like the deuterium in so-called heavy water.) Obviously,

$$\frac{T_2}{T_0} = \sin^2\psi\,, \tag{9.55}$$

in this limit.

9.7 A KINEMATICAL EXAMPLE IN THE LAB FRAME

Let's do an example. Let's say a particle of mass m_1 scatters elastically from one of mass m_2 at rest in the Lab frame. The ratio $\frac{v_1}{u_1} = f$ is given. Find the angle ψ through which m_1 is scattered. We have

$$v_1 = \frac{u_1}{(1+x)}\left[x\cos\psi \pm \sqrt{1 - x^2\sin^2\psi}\right].$$

Thus,

$$\Rightarrow f(1+x) = x\cos\psi \pm \sqrt{1 - x^2\sin^2\psi}$$

$$\Rightarrow f^2(1+x)^2 + x^2\cos^2\psi - 2f\,x(1+x)\cos\psi$$

$$= 1 - x^2\sin^2\psi$$

$$\Rightarrow f^2(1+x)^2 + x^2 - 2f(1+x)\,x\cos\psi = 1\,.$$

Solve for cos ψ in terms of x and f:

$$\cos\psi = \frac{x^2 - 1 + f^2(1+x)^2}{2f\,x(1+x)}\,.$$

To find the meaningful range for x, write

$$-1 \le \cos\psi \le 1$$

Plug $\cos\psi = 1$ into the above (note 0<f<1 and that x is positive):

$$\Rightarrow x(\cos\psi = 1) = \frac{f+1}{1-f}. \text{ (largest x)}$$

The other limit is for $\cos\psi = -1$, which we get by simply letting $f \to -f$. Thus

$$\Rightarrow x(\cos\psi = 1) = \frac{1}{x(\cos\psi = -1)} \text{ (smallest x)}.$$

9.8 RUTHERFORD SCATTERING IN THE LAB FRAME

Let's find the differential cross sections in the Lab frame. Remember the meaning of $d\sigma$:

$$d\sigma = \frac{dN}{I} = 2\pi\, b\, |db|\,. \tag{9.56}$$

This is unchanged going from one frame to another. (At this stage there is no reference to angles.) Then

$$b = b(\theta)\,,\ \text{CM frame} \quad \text{or} \quad b = b(\psi)\,,\ \text{Lab frame.}$$

The connection between Lab and CM cross sections is,

$$\Rightarrow \underset{\text{Lab}}{\underbrace{\frac{d\sigma}{d\Omega}(\psi)}} = \underset{\text{CM}}{\underbrace{\left(\frac{d\sigma}{d\Omega}(\theta)\right)}} \frac{d\Omega(\theta)}{d\Omega(\psi)} \tag{9.57}$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta)\,\frac{\sin\theta d\theta}{\sin\psi d\psi} \tag{9.58}$$

Result of "Aside 1":

$$\frac{\sin(\theta-\psi)}{\sin\psi} = x\,. \qquad \text{Do } \frac{d}{d\psi} \text{ on both sides:}$$

$$-\frac{\sin(\theta-\psi)}{\sin^2\psi}\cos\psi + \frac{\cos(\theta-\psi)}{\sin\psi}\left(\frac{d\theta}{d\psi} - 1\right) = 0\,,$$

$$\Rightarrow \frac{d\theta}{d\psi} - 1 = \frac{\sin(\theta-\psi)\cos\psi}{\cos(\theta-\psi)\sin\psi}\,. \tag{9.59}$$

Use

$$\sin(\theta-\psi) = x\sin\psi\,, \qquad \text{"Aside 1" again}$$

$$\cos(\theta-\psi) = \underset{\substack{\uparrow \\ \text{specializing to } \underline{x<1}.}}{+}\sqrt{1-x^2\sin^2\psi}\,, \qquad \text{"Aside 3".}$$

We find

$$\frac{d\theta}{d\psi} = 1 + \frac{x \sin\psi \cos\psi}{\sqrt{1 - x^2 \sin^2\psi}\ \sin\psi}, \tag{9.60}$$

$$\Rightarrow \frac{d\theta}{d\psi} = \frac{x \cos\psi + \sqrt{1 - x^2 \sin^2\psi}}{\sqrt{1 - x^2 \sin^2\psi}}. \tag{9.61}$$

so ($x < 1$ case; using "Aside 2" now in (9.58):

$$\frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta) \quad \frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi},$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta(\psi)) \frac{\left[x\cos\psi + \sqrt{1 - x^2\sin^2\psi}\right]^2}{\sqrt{1 - x^2\sin^2\psi}}. \tag{9.62}$$

Also from $\frac{\sin(\theta - \psi)}{\sin\psi} = x,$

$$\Rightarrow \theta = \psi + \sin^{-1}(x \sin \psi) \tag{9.63}$$

Notice for "b" large enough in the $x = \frac{m_1}{m_2} = \frac{V}{v_1'} > 1$ case, we are in the "stretch" case. When b is smaller than the value that causes scattering into angle $\psi_{MAX} = \sin^{-1}\left(\frac{m_2}{m_1}\right)$, then we are in the "short" case. (Formally, the critical value of the impact parameter is given by $b_0 = b\left(\psi = \sin^{-1}\left(\frac{m_2}{m_1}\right)\right).$) Both, however, cause scattering into the lab angle y, and thus the analog for x > 1 is the rather complicated expression

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta(\psi))\Big|_{stretch} \frac{\left[x\cos\psi + \sqrt{1 - x^2\sin^2\psi}\right]^2}{\sqrt{1 - x^2\sin^2\psi}}$$
$$+ \frac{d\sigma}{d\Omega}(\theta(\psi))\Big|_{short} \frac{\left[x\cos\psi - \sqrt{1 - x^2\sin^2\psi}\right]^2}{\sqrt{1 - x^2\sin^2\psi}}. \tag{9.64}$$

If x < 1, can do an expansion in powers of x. To $0^{\underline{th}}$ order,

$$\theta \approx \psi,$$

$$\frac{d\sigma}{d\Omega}(\psi) \simeq \frac{d\sigma}{d\Omega}(\theta)\Big|_{\theta=\psi},$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) \simeq \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\psi}{2}}. \tag{9.65}$$

Remember, p_∞ is measured in the CM frame. Connection with other quantities:

$$p_\infty = \mu\left|\dot{\vec{r}}_{initial}\right| = \mu\left(u_1' + u_2'\right),$$

$$m_1 u_1' = m_2 u_2' \Rightarrow u_2' = \frac{m_1}{m_2} u_1',$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

$$\Rightarrow p_\infty = \frac{m_1 m_2}{m_1 + m_2}\left(1 + \frac{m_1}{m_2}\right)u_1' = m_1 u_1' = \mu u_1 .$$

$$\uparrow$$

$$u_1' = \frac{u_1}{\left(1 + \frac{m_1}{m_2}\right)}$$

Now

$$\frac{p_\infty^2}{2\mu} = \frac{\mu^2 u_1^2}{2\mu} = \frac{\mu u_1^2}{2} = T_0',$$

But

$$T_0' = \frac{m_2}{m_1 + m_2} T_0 = \frac{1}{1 + x} T_0$$

Therefore for x << 1 we have

$$\frac{d\sigma}{d\Omega}(\psi) \approx \frac{1}{16}\left(\frac{k}{T_0}\right)^2 \frac{1}{\sin^4 \frac{\psi}{2}} . \tag{9.66}$$

written entirely in terms of Lab quantities. Now let's see what the first order correction to this expression (in x) is.

$$\theta = \theta_0 + x\theta_1 + \ldots ,$$

$$(9.63) \qquad \Rightarrow \theta_0 + x\theta_1 = \sin^{-1}(x \sin\psi) + \psi ,$$

$$\sin^{-1} y = y + \frac{y^3}{6} + \frac{3}{40} y^5 + \ldots , |y| < 1 ,$$

$$\Rightarrow \theta_0 + x\theta_1 \approx x \sin\psi + \psi ,$$

$$\Rightarrow \begin{cases} \theta_0 = \psi \\ \theta_1 = \sin\psi. \end{cases}$$

$$\frac{\left[x\cos\psi + \sqrt{1 - x^2\sin^2\psi}\right]^2}{\sqrt{1 - x^2\sin^2\psi}} \approx \frac{\left[1 + x\cos\psi\right]^2}{1} \approx 1 + 2x\cos\psi\,,$$

$$\sin\frac{\theta}{2} \simeq \sin\left(\frac{\psi}{2} + \frac{x}{2}\sin\psi\right),$$

$$\approx \sin\frac{4}{2} + \frac{x}{2}\sin\psi\cos\frac{\psi}{2}\,,$$

$$= \sin\frac{4}{2} + x\sin\frac{\psi}{2}\cos^2\frac{\psi}{2}\,,$$

$$= \sin\frac{4}{2}\left(1 + x\cos^2\frac{\psi}{2}\right),$$

$$\Rightarrow \sin^4\frac{\theta}{2} \simeq \sin^4\frac{\psi}{2}\left(1 + 4x\cos^2\frac{\psi}{2}\right).$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\psi) \approx \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\psi}{2}} \underbrace{\frac{(1 + 2x\cos\psi)}{\left(1 + 4x\cos^2\frac{\psi}{2}\right)}}\,,$$

$$\approx 1 + x\left(2\cos\psi - 4\cos^2\frac{\psi}{2}\right),$$

$$= 1 + x\left(-2\cos^2\frac{\psi}{2} - 2\sin^2\frac{\psi}{2}\right) = 1 - 2x\,.$$

Also

$$\frac{\mu k}{p_\infty^2} = \frac{k}{2\left(\frac{p_\infty^2}{2\mu}\right)} = \frac{k}{2T_0'} = \frac{k\,(1 + x)}{2T_0}\,,$$

$$\Rightarrow \left(\frac{\mu k}{p_\infty^2}\right)^2 \simeq \left(\frac{k}{2T_0}\right)^2(1 + 2x)\,.$$

Finally, then to first order in x:

$$\frac{d\sigma}{d\Omega}(\psi) \approx \frac{1}{16}\left(\frac{k}{T_0}\right)^2 \frac{1}{\sin^4\frac{\psi}{2}} \underbrace{(1+2x)(1-2x)}_{\simeq 1}.$$

First order correction vanishes! This means the Rutherford cross section, which is strictly an x=0 result, holds very accurately for 0 < x < 1. Will not torture you with the next order, ~x2, correction.

9.9 TOTAL CROSS SECTION

Total cross section:

$$\sigma \equiv \int_{\substack{\text{all}\\ \text{angles}}} d\sigma = \int \frac{d\sigma}{d\Omega}(\theta)\, d\Omega\,. \qquad (9.67)$$

$$(d\Omega = \sin\theta d\theta d\phi)$$

For Rutherford:

$$\sigma = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \int_0^{2\pi} d\phi \int_{0^+}^{\pi} \frac{\overbrace{\sin\theta}^{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\, d\theta}{\sin^4\frac{\theta}{2}} .$$

(strictly speaking, the point $\theta = 0$ is excluded in the integration)

$$\sigma = 2\pi\left(\frac{\mu k}{p_\infty^2}\right)^2 \int_{0^+}^{\pi} \frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} \frac{d\theta}{2},$$

$$\sigma = 3\pi\left(\frac{\mu k}{p_\infty^2}\right)^2 \left[-\frac{1}{2\sin^2\frac{\theta}{2}}\right]\Bigg|_{0^+}^{\pi} \rightarrow +\infty.$$

Understandable: particles are *always* deflected regardless of b value. Can see it in:

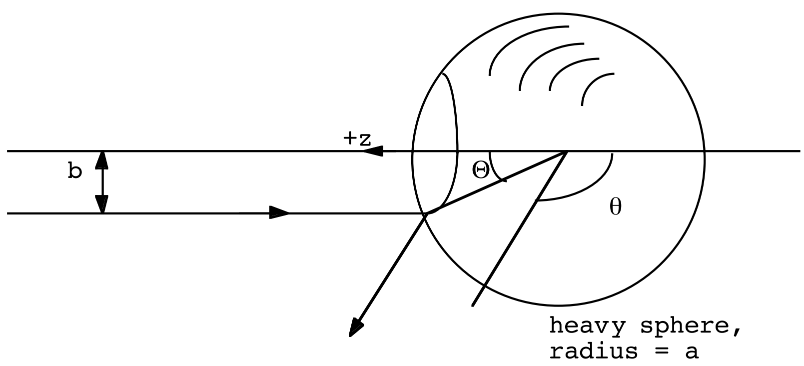
$$d\sigma = 2\pi\, b db ,$$

$$\Rightarrow \sigma = 2\pi \int_0^{b_{max}} b\, db = \pi b_{max}^2 . \tag{9.68}$$

where b_{max} is the maximum impact parameter which suffers an angular deflection $\neq 0$. Thus, in classical mechanics, the only type of force laws for which σ is finite are those of the form

$$F(r) = 0 \quad , \quad r > a$$

where a is some finite value of separation. Example of this possibility ($\theta = \psi$ here since the sphere is infinitely heavy):



Clearly, this problem has azimuthal symmetry if we take the +z axis as shown. We have

$$b = a \sin \Theta ,$$

$$2\Theta + \theta = \pi .$$

notice: -sign
(repulsive scattering)
↓

$$\Theta = \frac{\pi - \theta}{2} ,$$

$$b = a \sin\left(\frac{\pi - \theta}{2}\right) = a \cos\frac{\theta}{2} .$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \quad ; \quad \frac{db}{d\theta} = -\frac{a}{2}\sin\frac{\theta}{2} ;$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{a \cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \cdot \frac{a}{2}\sin\frac{\theta}{2} = \frac{1}{4}a^2 .$$

$$\Rightarrow \sigma = \int \frac{1}{4}a^2\, d\Omega = \pi a^2 .$$

Just what we expect.

9.10 PROBLEMS

1. The Rutherford differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}.$$

Therefore, the backscattering cross section is

$$\sigma_{\text{back}} = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta \frac{\sin\theta}{\sin^4\frac{\theta}{2}}.$$

In the text I estimated the integral by setting $\left.\frac{1}{\sin^4\frac{\theta}{2}}\right|_{\theta=\pi} = 1$, which resulted in an underestimate of σ_{back}. Now do this integral exactly. By what factor was the original estimate off? Will this improve the agreement with Rutherford's experiment?

2. We estimated in the notes the fraction of 5 MeV alpha particles backscattered ($\frac{\pi}{2} < \theta < \pi$) from a target made of gold foil and found that about one in every 21,000 particles should backscatter. (This was changed to about one in every 10,500 after we corrected the integral.) Suppose Rutherford wanted to make a lead shield to protect the other experiments in his lab. Note that for lead, Z = 82 and m_{Pb} = 207 amu, and that the density is $\rho_{Pb} = 11.4\,\frac{gm}{cm^3}$. Estimate the minimum thickness of the lead that would shield against these alpha particles (in centimeters). [Hint: Require all particles to be backscattered from the lead. Neglect the possibility of multiple scattering.]

3. Given that (T_0' is total CM energy)

$$b = \frac{k}{\left(T_0'\right)^2}\frac{1}{\theta},$$

exactly for some unknown central force law, find

a) the CM differential cross section, $\frac{d\sigma}{d\Omega}(\theta)$.
b) the number of particles backscattered (that is, with angles θ such that $\frac{\pi}{2} < \theta < \pi$) in the CM frame. (Call this N_{back}). Assume a known incoming particle flux, I.

4. Do the integration in (9.16) leading to (9.17) of the notes.

5. Consider scattering off of a weak potential U(r) such that U(b)<< E, where b is the impact parameter and E is the total energy. Show that

$$r_{min} \approx b\left(1 + \frac{U(b)}{2E}\right),$$

which shows that r_{min} < b for an attractive potential (U(b)<0)and vice versa, as one would expect.

6. Look up and plug values in (M_S, R_S are the Sun's mass, radius)

$$\Delta\theta = \frac{4GM_S}{c^2}\frac{1}{R_S},$$

to get the angular deflection for starlight in *seconds of arc*.

7. Consider a head-on collision as seen from an unknown reference frame. Assume the ratio $x = \frac{m_1}{m_2}$ is known. It is observed that m_1 comes to a complete stop after the interaction. Assuming energy and momentum conservation, find the after/before ratio of m_2's kinetic energies in terms of x.

8. Derive:

(a) $$\tan\psi = \frac{\sin 2\xi}{x - \cos 2\xi},$$

(b) $$\sin\psi = \frac{\sin\phi}{\sqrt{1 + x^2 - 2x\cos\phi}},$$

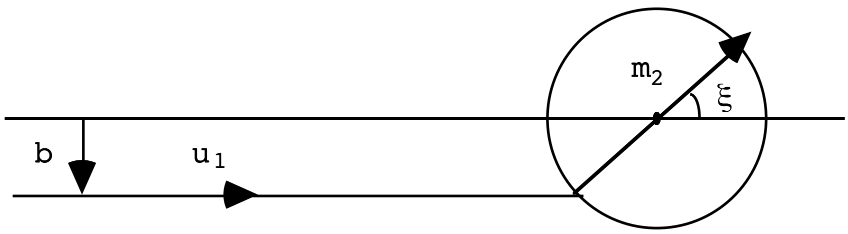
where $x = \frac{m_1}{m_2}$ and the angles ψ and ϕ are defined in the text.

9. From prob. 9.8(b), or any other means, show that

$$\tan\psi = \frac{\sin\phi}{x - \cos\phi}.$$

10. In reference to Eq.(9.65) of the text, evaluate the critical impact parameter variable, b_0, for the Coulomb scattering potential, $U(r) = \pm\frac{k}{r}$, in terms of k, μ, p_∞ and x.

11. Consider scattering of a point mass m1 off of a hard sphere of radius a and mass m_2. (The ratio $x = \frac{m_1}{m_2}$ is arbitrary.)



The angle of the sphere's (m2's) deflection in the laboratory frame (m2 initially stationary) is given by , $\xi = \sin^{-1}\left(\frac{b}{a}\right)$,where b is the impact parameter.

a) Show that the cross section evaluated in the CM frame of reference is a constant.
b) Find the deflection angles θ and ψ for m1 if the impact parameter is $b = \frac{a}{\sqrt{2}}$.

In addition to the above, find the the cross section in the Lab frame of reference for c) x<1 and d) x>1.

12. Show that (9.62) of the notes can be written more simply as

$$\frac{d\sigma}{d\Omega}(\psi) = \frac{d\sigma}{d\Omega}(\theta)\left(\frac{v_1}{v_1'}\right)^2 \frac{v_1 v_1'}{\vec{v}_1 \cdot \vec{v}_1'}.$$

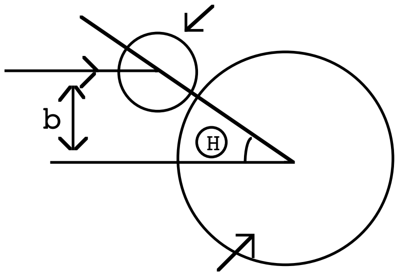
13. Given a center of mass differential cross section,

$$\frac{d\sigma}{d\Omega} = A\cos^2\theta\,,$$

(θ is the deflection angle of m1 in the CM frame) and a particle flux, I, find the number of backscattered particles ($\frac{\pi}{2} < \psi < \pi$) per unit time in the Lab frame. Assume $m_2 >> m_1$ where m1 is the mass of the incident particle.

14. With the same cross section as in problem 9.13 above, find the full Lab cross section for $x < 1$. How does one specify the "short" case in the relation between θ and ψ?

15. Hard sphere scattering:



Assuming a symmetrical scattering event, find $\frac{d\sigma}{d\Omega}$ and the total cross section.

Other Problems

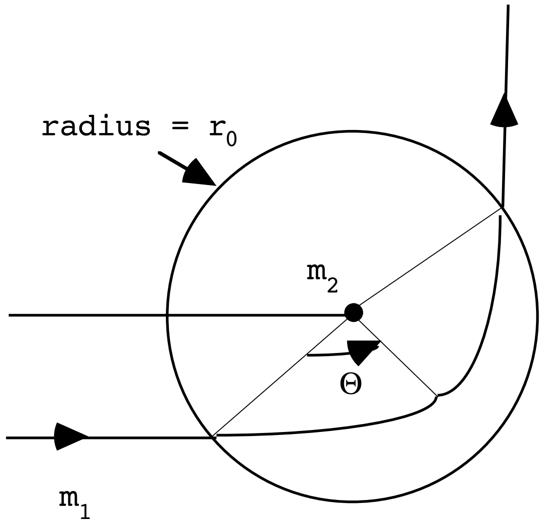
16. Escape velocity is the minimum speed a particle needs to escape a planet or star, starting from it's surface. Previously, we estimated the angular deflection of a light beam traveling near the Sun, treating the light as if it were an ordinary massive particle. In the same spirit, find a formula for the maximum radius of a star of mass M from which light, traveling at the speed of light, c, may no longer escape. (This is called the "Schwartzchild radius" of the star.)

17. The maximum scattering angle, ψ_{max}, in the Lab frame for m_1 in the $x > 1$ case ($x = \frac{m_1}{m_2}$) was given by $\sin(\psi_{max}) = \frac{1}{x}$. Show that this happens when the CM scattering angle, θ, satisfies,

$$\cos\theta = -\frac{1}{x}.$$

18. Consider a finite range inverse square force law. That is, we have an attractive force

$$F(r) = \begin{cases} -\frac{k}{r^2}, & r < r_0 \\ 0, & r > r_0 \end{cases}$$



Show that the deflection angle Θ is given by (E=total energy,b=impact parameter)

$$\cos\Theta = \frac{2b^2 - \dfrac{kr_0}{E}}{r_0\sqrt{\left(\dfrac{k}{E}\right)^2 + 4b^2\left(1 - \dfrac{k}{r_0 E}\right)}}\,.$$

19. Concerning the same problem as 9.18, show that the CM scatterimg angle θ can be written as

$$\theta = 2\sin^{-1}\left(\frac{b}{r_0}\right) - \sin^{-1}(\cos\Theta)\,.$$

20. Given the results in probs. 9.18 and 9.19, go as far as possible in finding the scattering cross section, $\frac{d\sigma}{d\Omega}$, in the CM frame for this force law.

21. Eq.(9.62) gives the scattering cross section in the Lab frame, $\frac{d\sigma}{d\Omega}(\psi)\,|_{Lab}$ in terms of the cross section in the CM frame, $\frac{d\sigma}{d\Omega}(\theta(\psi))\,|_{CM}$, for the $x = m_1/m_2 < 1$ case. Turn this around by expressing $\frac{d\sigma}{d\Omega}(\theta)\,|_{CM}$ in terms of the laboratory quantity, also for x<1. [Ans.:

$$\frac{d\sigma}{d\Omega}(\theta)\bigg|_{CM} = \frac{d\sigma}{d\Omega}(\psi(\theta))\bigg|_{Lab} \cdot \frac{(1 + x\cos\theta)}{[1 + x^2 + 2x\cos\theta]^{3/2}},$$

where

$$\tan\psi = \frac{\sin\theta}{\cos\theta + x}.]$$

22. (a) Show directly that the equations ($x = m_1/m_2$)

$$\theta = \psi + \sin^{-1}(x\sin\psi),$$

and

$$\cos\theta = -x\sin^2\psi \pm x\cos\psi\sqrt{\frac{1}{x^2} - \sin^2\psi},$$

are compatible.

(b) For $x = 2$, what two roots, θ_1, θ_2, are associated with $\psi = \pi/8$?

23. Given the result for the Rutherford differential cross section in the CM frame for m_1,

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} \quad ,$$

find the differential cross section **for m_2** (NOT m_1) as a function of the appropriate angle:

a) in the center of mass (CM) frame
b) in the Laboratory frame (**m_2 initially at rest**)

24. Given the result for the Rutherford differential cross section in the CM frame (all for m_1),

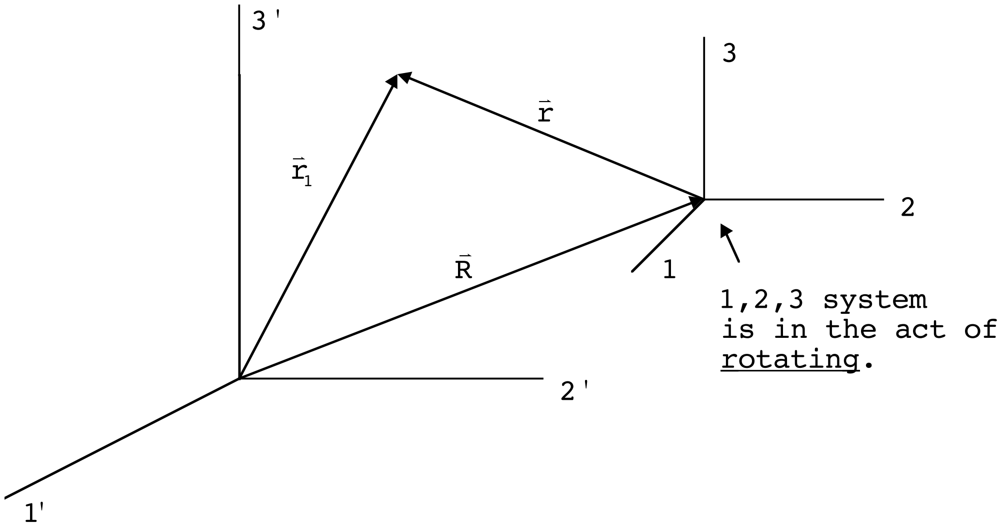
$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{4}\left(\frac{\mu k}{p_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} \quad ,$$

find the differential cross section, $\frac{d\sigma}{d\Omega}(\psi)$, in the Laboratory frame, when $m_1 = m_2$ ($x = 1$).

10 NON INERTIAL REFERENCE FRAMES

10.1 FINITE DISPLACEMENTS AND ROTATIONS

Need transformation connecting fixed and moving (body) noninertial frames. Picture:



Going to let the 1,2,3, system "evolve" an infinitesimal amount in time, dt	⇒	**We are going to consider an <u>infinitesimal</u> relation for the $\vec{r}$ axes, but <u>finite</u> displacements, $\vec{R}$**

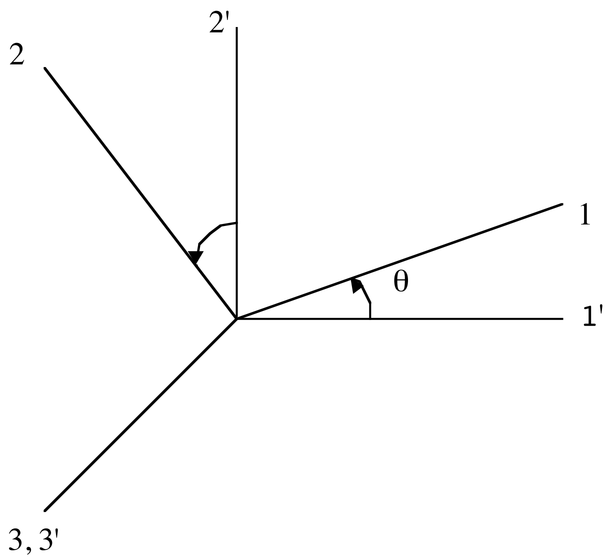
(1,1'), (2,2'), (3,3') axes coincide in direction at some instant of time, t. Clearly, the reason for this is to describe, for example, motions relative to the Earth, which is noninertial. This is all done for convenience, not any real physics reason. In fact, strictly speaking, this chapter has zero physics content! This does not mean, however, that these considerations are not useful or convenient.

There are two types of transformations that will be involved:

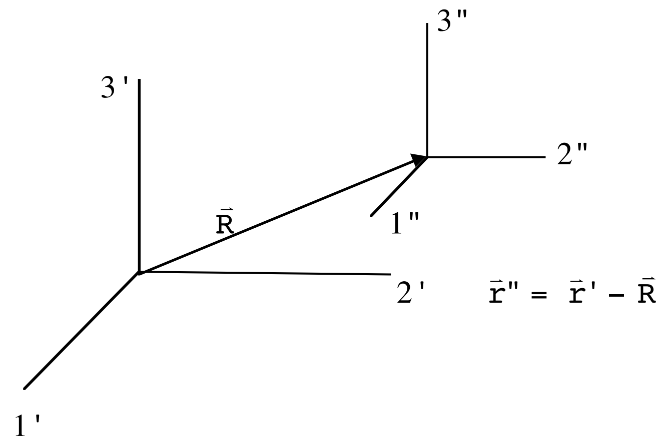
1. Displacement: $\vec{r} = \vec{r}\,' - \vec{R}$. (see above figure)
2. Rotation: (a "generic" passive rotation)

$$r_i = \sum_j \lambda_{ij} r_j'. \qquad \text{(passive)}$$

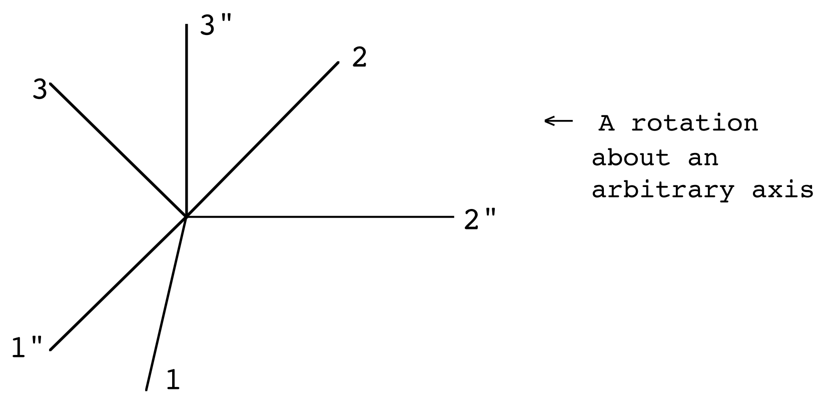
Picture: (specialized to rotation about 3,3')



Put them together. Step 1: displacement. At the end of step 1:



Now wish to rotate. Rotate the $\vec{r}\,''$ axes:

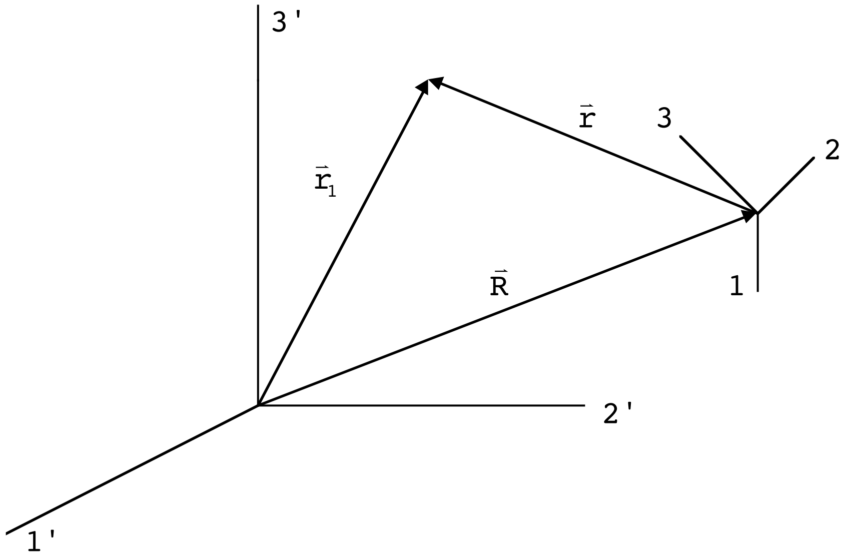


$$r_i = \sum \lambda_{ij} r_j'' . \tag{10.1}$$

But $r''_j = r'_j - R_j$, so

$$r_i = \sum_j \lambda_{ij} \left(r_j' - R_j \right). \tag{10.2}$$

Represents:



In matrix notation, this is

$$r = \lambda(r' - R). \tag{10.3}$$

It's inverse is

$$r' - R = \lambda^{-1} r = \lambda^{T} r. \tag{10.4}$$

We really need only the relationships between $\vec{r}$ and $\vec{r}'$ for an *infinitesimal* rotation. What is λ for such a situation? Remember

$$\lambda^{-1}\lambda = 1 \quad , \quad \lambda^{-1} = \lambda^{T},$$

$$\sum_i \lambda_{ij}\, \lambda_{ik} = \delta_{jk}. \tag{10.5}$$

Assume

$$\lambda_{ij} = \delta_{ij} + \underset{\uparrow}{\delta\lambda_{ij}}. \tag{10.6}$$

change in λ necessary
to represent an infinitesimal
passive rotation

Substitute above:

$$\sum \left(\delta_{ij} + \delta\lambda_{ij}\right)\left(\delta_{ik} + \delta_{ik}\right) = \delta_{jk} \, . \tag{10.7}$$

$$0^{\underline{th}} \text{ order: } \sum_i \delta_{ij}\delta_{ik} = \delta_{ik} \, . \quad \surd$$

$$1^{\underline{st}} \text{ order: } \sum_i \left(\delta_{ij}\,\delta\lambda_{ik} + \delta\lambda_{ij}\delta_{ik}\right) = 0 \, ,$$

$$\Rightarrow \delta\lambda_{ij} + \delta\lambda_{kj} = 0 \, .$$

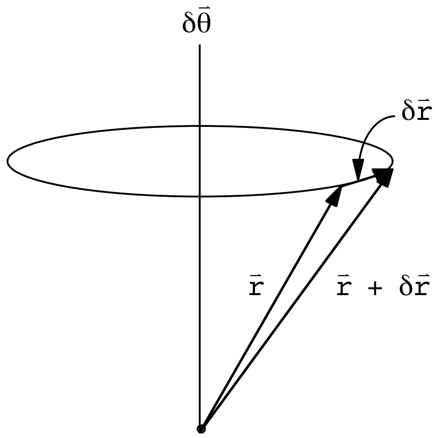
$$\text{or } \delta\lambda_{jk} = -\delta\lambda_{kj} \, , \; \underline{\text{antisymmetric}} \tag{10.8}$$

Also implies there are only 3 *independent* elements:

$$\delta\lambda_{ij} = \begin{pmatrix} 0 & (1) & (2) \\ -(1) & 0 & (3) \\ -(2) & -(3) & 0 \end{pmatrix} \qquad (1)\,,(2),(3) \text{ arbitrary elements}$$

10.2 INSTANTANEOUS RELATIONS FOR VELOCITY, ACCELERATION

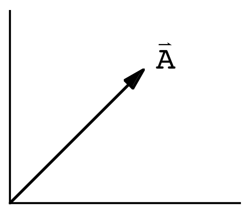
Now go back to Ch.1. Representation of an infinitesimal *active* rotation on a vector:



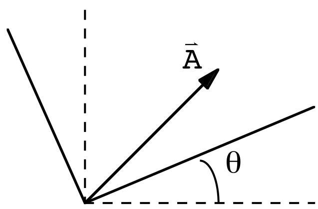
$$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}\,,$$

$$\text{or}\quad \delta r_i = \sum_{j,k} \varepsilon_{ijk}\, \delta\theta_j\, r_k\,. \tag{10.9}$$

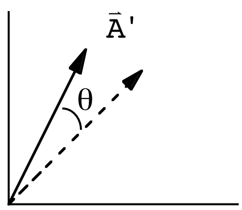
Now, any active rotation is given by a passive rotation in the opposite direction. Start:



Passive, – θ:
(λ_{ij})



Active,θ:

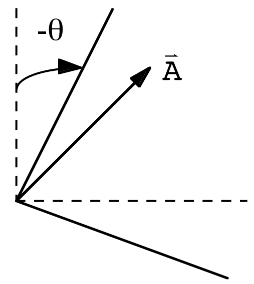


Passive , – θ
$\left(\lambda_{ij}^{-1} = \lambda_{ij}^{T}\right)$

↑

have the same
components

↓



This gives us two ways of specifying the effects of an active rotation:

Way 1: $\delta r_i = \sum_{j,k} \varepsilon_{ijk}\, \delta\theta_j\, r_k\,.$ (active rotation)

Way 2: $\delta r_i = \sum_j \lambda_{ij}^{T}\, r_j - r_i\,,$ (passive inverse rotation)

(final) (initial)

$$\text{or} \qquad \delta r_i = \sum_j \lambda_{ij}^{T}\, r_j - r_i\,, \tag{10.10}$$

Must be the same:

$$\sum_k \delta\lambda_{ik}^{T}\, r_k = \sum_{j,k} \varepsilon_{ijk}\, \delta\theta_j\, r_k\,,$$

$$\Rightarrow \delta\lambda_{ik}^{T} = \sum_j \varepsilon_{ijk}\, \delta\theta_j\,,$$

$$\text{or} \qquad \Rightarrow \delta\lambda_{ki} = \sum_j \varepsilon_{kij}\, \delta\theta_j\,. \tag{10.11}$$

Notice, as expected, $\delta\lambda_{ki}$ is antisymmetric in k,i.

Our relationship between the primed and unprimed coordinates are again,

$$r_i' - R_i = \sum_j \lambda_{ij}^{T}\, r_j\,. \tag{10.12}$$

Consider an infinitesimal change on both sides:

$$\delta r_i' - \delta R_i = \overset{(1)}{\sum_j \delta\lambda^T_{ij}\, r_j} + \overset{(2)}{\sum_j \lambda^T_{ij}\, \delta r_j}\,. \tag{10.13}$$

Why 2 terms on right side? (1) arises from the rotation of the noninertial axes while (2) is due to the independent motion of the particle relative to the $\vec{r}$ axes. Found earlier,

$$\delta\lambda^T_{ij} = \sum_k \varepsilon_{ikj}\, \delta\theta_k\,.$$

Of course also

$$\lambda^T_{ij} = \delta_{ij} + \delta\lambda^T_{ij}, \tag{10.14}$$

so

$$\delta r_i' - \delta R_i = \sum_{k,j} \varepsilon_{ikj}\delta\theta_k r_j + \sum_j \left(\delta_{ij} + \underset{\uparrow}{\delta\lambda_{ij}^T}\right)\delta r_j \,. \tag{10.15}$$

can drop (2nd order in small quantities)

Thus

$$\delta r_i' - \delta R_i = \sum_{k,j} \varepsilon_{ikj}\, \delta\theta_k r_j + \delta r_i \,. \tag{10.16}$$

Divide by δt $\left(\frac{\delta Q}{\delta t} \equiv \frac{dQ}{dt}\right)$:

$$\Rightarrow \frac{dr_i'}{dt} - \frac{dR_i}{dt} = \frac{dr_i}{dt} + \sum_{k,j} \varepsilon_{ikj} \frac{d\theta_k}{dt} r_j \,. \tag{10.17}$$

But

$$\omega_k \equiv \frac{d\theta_k}{dt}, \tag{10.18}$$

so

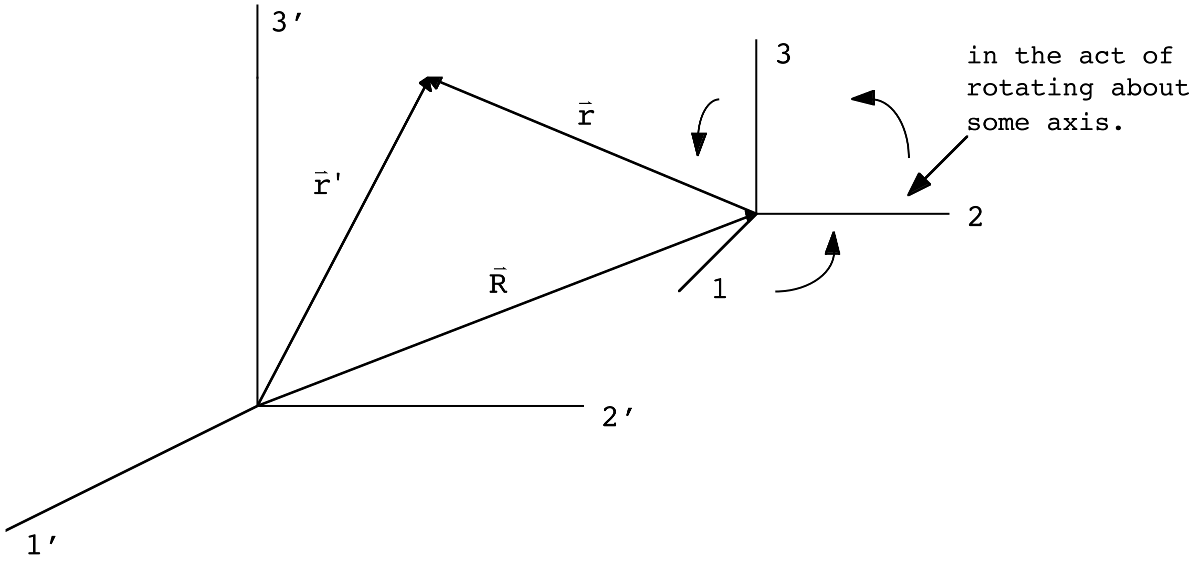
$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{R}}{dt} + \vec{\omega} \times \vec{r}, \tag{10.19}$$

or

$$\vec{v}_f = \vec{v}_r + \vec{V} + \vec{\omega} \times \vec{r} \,. \tag{10.20}$$

$\vec{v}_f$ $\left(= \frac{d\vec{r}\,'}{dt}\right)$ is the velocity of the particle relative to the fixed frame. (It is the velocity as measured by an observer at rest in $\vec{r}\,'$) Must be a constant in magnitude and direction if the particle has no real forces acting on it. $\vec{v}_r$ is the velocity relative to the moving frame, whose axes coincide with the fixed aces at the given instant in time.

Picture:



Now do the second variation:

$$\delta r_i' - \delta R_i = \sum_j \delta\, \lambda^T_{ij}\, r_j + \sum_j \lambda^T_{ij}\, \delta r_j\,,$$

$$\Rightarrow\ \delta^2 r_i' - \delta^2 R_i = \sum \delta^2\, \lambda^T_{ij}\, r_j + 2\sum_j \delta\lambda^T_{ij}\, \delta r_j$$

$$+ \sum_j \underset{\substack{\uparrow \\ \text{can replace by } \delta_{ij}}}{\lambda^T_{ij}}\, \delta^2\, r_j\,. \tag{10.21}$$

Before, we compared

$$\delta r_i = \varepsilon_{ijk}, \tag{10.22}$$

to

$$\delta r_i = \sum_k \delta\lambda^T_{ik}\, r_k\,, \tag{10.23}$$

and got

$$\delta\lambda^T_{ik} = \sum_j \varepsilon_{ijk}\delta\theta_j\,. \tag{10.24}$$

Now

$$\delta^2 r_i = \sum_{j,k} \varepsilon_{ijk}\, \delta\left(\delta\theta_j r_k\right)$$

$$= \sum_{j,k} \varepsilon_{ijk} \left(\delta^2\theta_j r_k + \delta\theta_j\, r_k\right). \tag{10.25}$$

But

$$\delta r_k = \sum_{\ell,m} \varepsilon_{k\ell m}\, \delta\theta_\ell r_m\,, \tag{10.26}$$

so

$$\delta^2 r_i = \sum_{j,k} \varepsilon_{ijk} \left(\delta^2\theta_j r_k + \sum_{\ell,m} \varepsilon_{k\ell m} \delta\theta_j \delta\theta_\ell r_m\right),$$

$$(\text{let } m \rightarrow k)$$

$$= \sum_{j,k} \varepsilon_{ijk}\, \delta^2\theta_j r_k + \sum_{\substack{j,k,\\ \ell,m}} \varepsilon_{ijm}\, \varepsilon_{m\ell k} \delta\theta_j\, \delta\theta_\ell r_k\,. \tag{10.27}$$

On the other hand, compare this to

$$\delta^2 r_i = \sum_k \delta^2 \lambda^T_{ik}\, r_k \,. \tag{10.28}$$

Identify:

$$\delta^2 \lambda^T_{ik} = \sum_j \varepsilon_{ijk}\, \delta^2 \theta_j + \sum_{j,\ell,m} \varepsilon_{ijm} \varepsilon_{m\ell k}\, \delta\theta_j \delta\theta_\ell \,. \tag{10.29}$$

(Switch indices: $k \to j$, $j \to k$ everywhere.) Put it all back together:

$$\delta^2 r_i' - \delta^2 R_i = \sum_{j,k} \varepsilon_{ikj} \delta^2 \theta_k r_j + \sum_{\substack{j,k,\\ \ell,m}} \varepsilon_{ikm}\, \varepsilon_{m\ell j} \delta\theta_k\, \delta\theta_\ell r_j$$
$$+ 2 \sum_{j,k} \varepsilon_{ikj} \delta\theta_k \delta r_j + \delta^2 r_i \,. \tag{10.30}$$

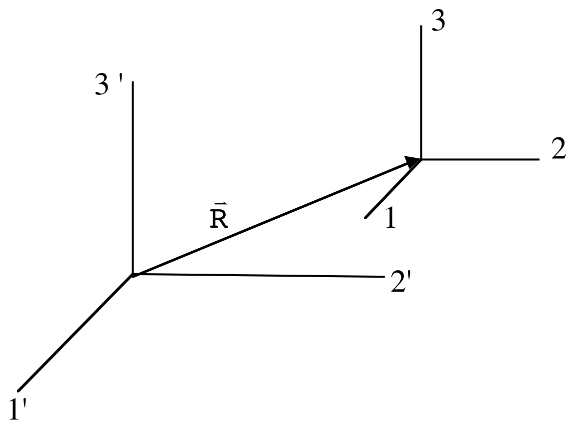
Divide by dt^2 $\left(\dfrac{\delta^2 Q}{\delta t^2} = \dfrac{d^2 Q}{dt^2}\right)$:

$$\Rightarrow \frac{d^2 r_i'}{dt^2} - \frac{d^2 R_i}{dt^2} = \sum_{k,j} \varepsilon_{ikj} \frac{d^2 Q_k}{dt^2} + \frac{d^2 r_i}{dt^2}$$
$$+ \sum_{k,m} \varepsilon_{ikm} \frac{d\theta_k}{dt} \left(\sum_{\ell,j} \varepsilon_{m\ell j} \frac{d\theta_\ell}{dt} r_j \right) + 2 \sum_{j,k} \varepsilon_{ikj} \frac{d\theta_k}{dt} \frac{dr_j}{dt} \,. \tag{10.31}$$

Identify $\omega_i = \dfrac{d\theta_i}{dt}$, $\dot{\omega}_i = \dfrac{d^2 \theta_i}{dt^2}$ and write in vector notation:

$$\ddot{\vec{r}}' = \ddot{\vec{R}} + \ddot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \dot{\vec{r}} \,. \tag{10.32}$$

$\ddot{\vec{R}}$ represents the acceleration of the *origin* of the $\vec{r}$ coordinate system relative to the $\vec{r}\,'$ origin. Will be zero if we consider uniform motion. Picture:



Also have $\dot{\vec{\omega}} = 0$ in the case of constant angular velocity (magnitude and direction).

10.3 USEFUL EARTH COORDINATE CHOICES

Write in the *inertial system*

$$\vec{F} = m\ddot{\vec{r}}, \tag{10.33}$$

Write in the *noninertial system*

$$\vec{F}_{eff} = m\ddot{\vec{r}}. \tag{10.34}$$

which gives

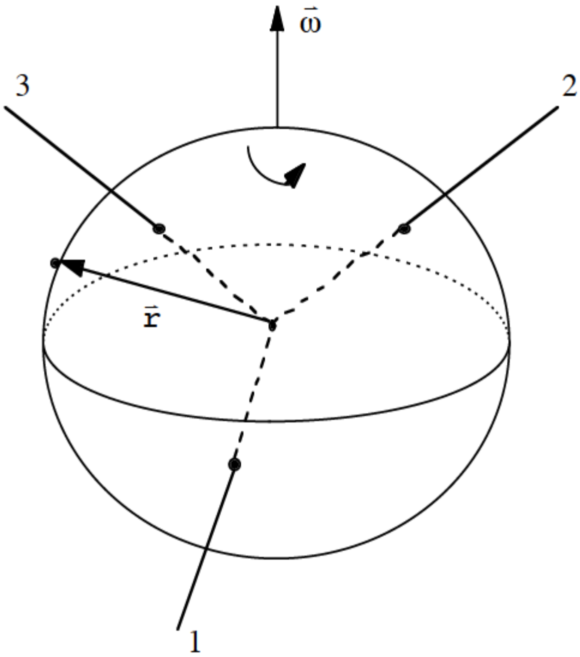
$$\vec{F}_{eff} = m\left(\ddot{\vec{r}}' - \ddot{\vec{R}} - \dot{\vec{\omega}} \times \vec{r} - \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\vec{\omega} \times \dot{\vec{r}}\right),$$

or

$$\vec{F}_{eff} = \vec{F} - m\left(\ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} - \underbrace{\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)}_{\text{``centifugal''}} + \underbrace{2\vec{\omega} \times \dot{\vec{r}}}_{\text{``Coriolis''}}\right). \tag{10.35}$$

Remember:		
Deflection is to the right in northern hemisphere and to the left in the southern (relative to the initial direction)	$\Rightarrow$	**Leads to deflection of air masses in a counter-clockwise direction in the northern hemisphere**

For the Earth, we often choose:



For this choice $\dot{\vec{\omega}} = \ddot{\vec{R}} = 0$, so

$$\vec{F}_{eff} = \vec{F} - m\left(\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \dot{\vec{r}}\right). \tag{10.36}$$

Sum up. Because instantaneously our axes coincide in direction, we have

$$\vec{r}' = \vec{r} + \vec{R},$$

$$\dot{\vec{r}}' = \dot{\vec{r}} + \dot{\vec{R}} + \vec{\omega} \times \vec{r},$$

$$\ddot{\vec{r}}' = \ddot{\vec{r}} + \ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right).$$

How come we do not get $\dot{\vec{r}}'$ as simply the time derivative of the expression $\vec{r}'$ for example? Because $\vec{r}'$ and $\vec{r}$ are referred to different axes which are rotating as well as moving with velocity $\dot{\vec{R}}$ with respect to one another. There is another way of viewing this process more in line with the book's derivation. When a change in $\vec{r}$ is considered, calculated with respect to the moving axes, we have

$$\underset{\substack{\uparrow \\ \text{fixed}}}{\left(\delta\vec{r}\right)_f} = \underset{\substack{\uparrow \\ \text{rotating}}}{\left(\delta\vec{r}\right)_r} - \left(\delta\vec{r}\right)_{passive}, \tag{10.37}$$

From before:

$$\left(\delta\vec{r}\right)_{passive} = -\left(\delta\vec{r}\right)_{active} = -\delta\vec{\theta} \times \vec{r}, \quad \xleftarrow{\text{understood in fixed frame}}$$

$$\Rightarrow \left(\delta\vec{r}\right)_f = \left(\delta\vec{r}\right)_r + \delta\vec{\theta} \times \vec{r}. \tag{10.38}$$

Thus

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r}. \qquad \left(\vec{\omega} = \frac{d\vec{\theta}}{dt}\right) \tag{10.39}$$

This is true for any vector, not just r, as long as the fixed and rotating axes instantaneously coincide in direction. Including the effect of translation of the coordinate origin, this now gives

$$\left(\frac{d\vec{r}'}{dt}\right)_f = \underbrace{\left(\frac{d\vec{r}}{dt}\right)_f}_{\substack{\uparrow \\ \text{make replacement}}} + \underbrace{\frac{d\vec{R}}{dt}}_{\substack{\uparrow \\ \text{understood in fixed frame}}}$$

$$\Rightarrow \dot{\vec{r}}' = \dot{\vec{r}} + \dot{\vec{R}} + \vec{\omega} \times \vec{r} \text{ as before.}$$

Apply the same reasoning to get $\ddot{\vec{r}}'$:

$$\left(\frac{d\dot{\vec{r}}'}{dt}\right)_f = \underline{\underline{\left(\frac{d\dot{\vec{r}}}{dt}\right)_f}} + \frac{d\dot{\vec{R}}}{dt} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \overset{\text{see above}}{\overset{\downarrow}{\underline{\underline{\left(\frac{d\vec{r}}{dt}\right)_f}}}}. \tag{10.40}$$

However

$$\left(\frac{d\dot{\vec{r}}}{dt}\right)_f = \left(\frac{d\dot{\vec{r}}}{dt}\right)_r + \vec{\omega} \times \dot{\vec{r}}, \tag{10.41}$$

$$\Rightarrow \ddot{\vec{r}}' = \ddot{\vec{r}} + \vec{\omega} \times \dot{\vec{r}} + \ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\dot{\vec{r}} + \vec{\omega} \times \vec{r}\right),$$

$$\Rightarrow \ddot{\vec{r}}' = \ddot{\vec{r}} + \ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right).$$

This is also as before.

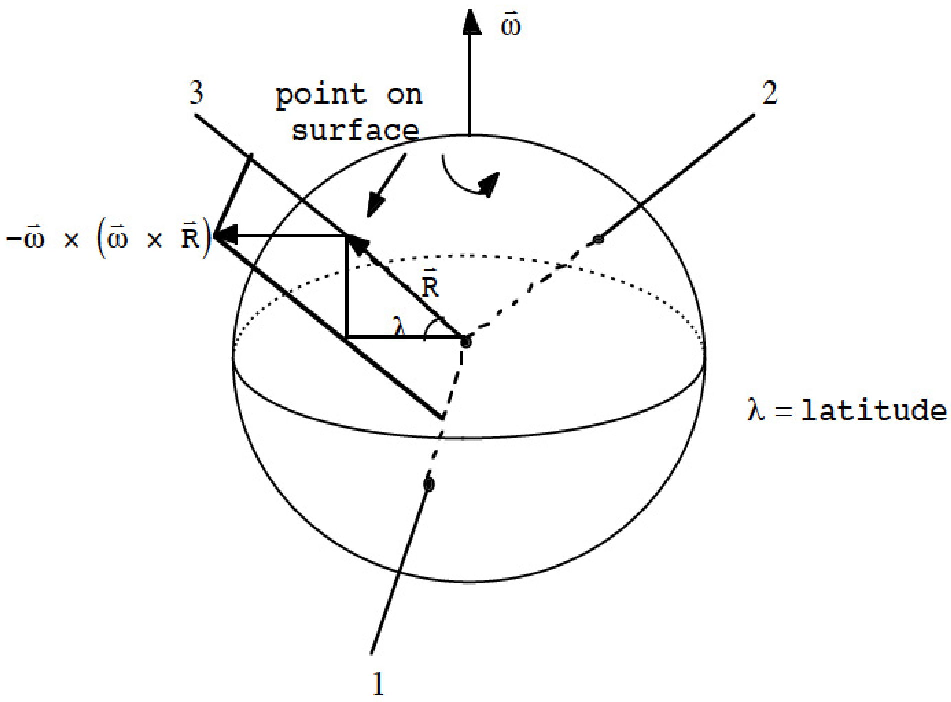
Get back to $\vec{F}_{eff}$ on Earth (static case, $\dot{\vec{r}} = 0$)

$$\vec{F}_{eff} = \vec{F} - m\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right). \tag{10.42}$$

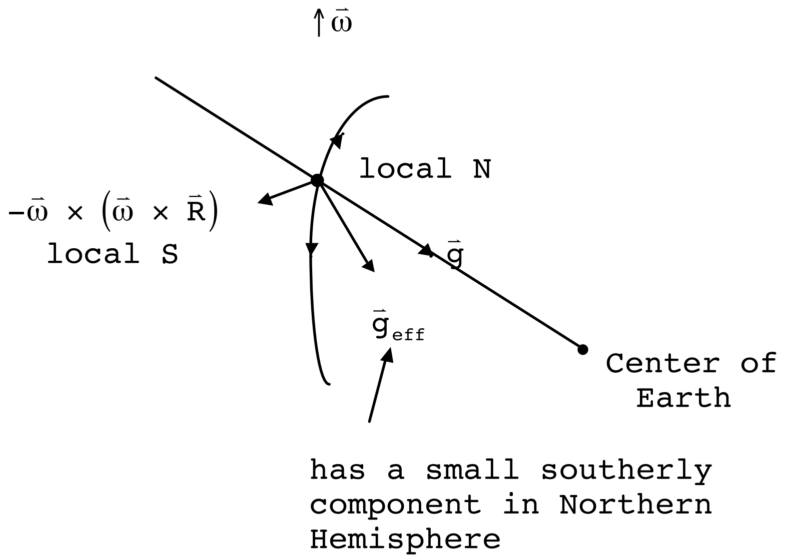
This means that the *effective* acceleration due to gravity and the Earth is given by

$$\vec{g}_{eff} = \underbrace{\vec{g}}_{\substack{\uparrow \\ \text{gravity only}}} - \vec{\omega} \times \left(\vec{\omega} \times \underbrace{\vec{R}}_{\substack{\uparrow \\ \vec{R} \text{ points from center of Earth to surface}}}\right). \tag{10.43}$$

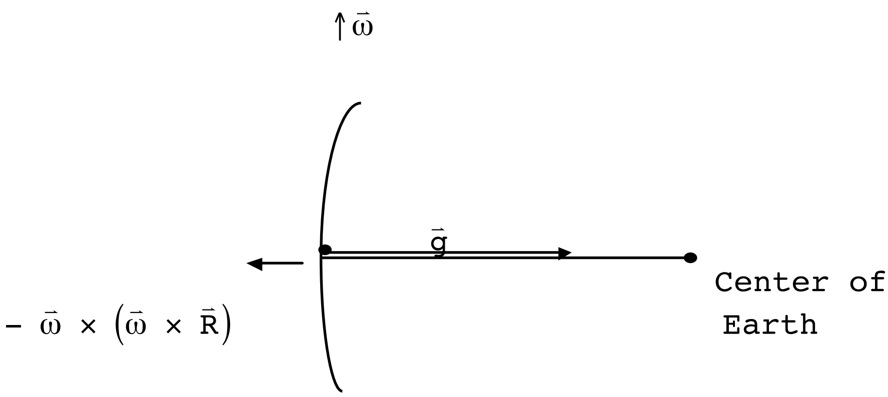
Picture:



As we can see, $-\vec{\omega} \times (\vec{\omega} \times \vec{R})$ has components along 3 and 1 axes. Another view (corresponds to the 13 plane above):

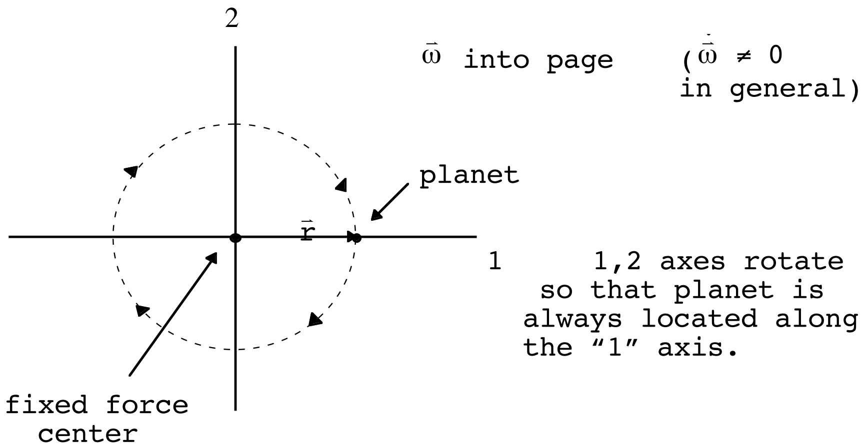


At equator, it's particularly simple:



You will find the angular deviation of a plumb line from the true vertical caused by this effect in a problem.

Example: central force problem.



$$\vec{F} = -\frac{dU}{dr}\hat{e}_1 \ , \ \vec{R} = 0$$

$$m\ddot{\vec{r}} = \vec{F} - m\left(\overset{0}{\cancel{\ddot{\vec{R}}}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\omega \times \dot{\vec{r}}\right),$$

$$\Rightarrow \begin{cases} \vec{\omega} = -\omega\hat{e}_3 \ , \ \dot{\vec{\omega}} = -\dot{\omega}\hat{e}_3, \\ \vec{r} = r\hat{e}_1 \ , \dot{\vec{r}} = \dot{r}\hat{e}_1 \ , \quad \left(\dot{\hat{e}}_1 = 0 \text{ in rotating frame}\right) \\ \ddot{\vec{r}} = \ddot{r}\hat{e}_1, \\ \dot{\vec{\omega}} \times \vec{r} = -\dot{\omega}r\hat{e}_2, \\ 2\vec{\omega} \times \dot{\vec{r}} = -2\omega\dot{r}\hat{e}_2, \\ \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) = -\omega^2 r\hat{e}_1. \end{cases}$$

"2" components:

$$0 = -m\left(-\dot{\omega}r - 2\omega\dot{r}\right).$$

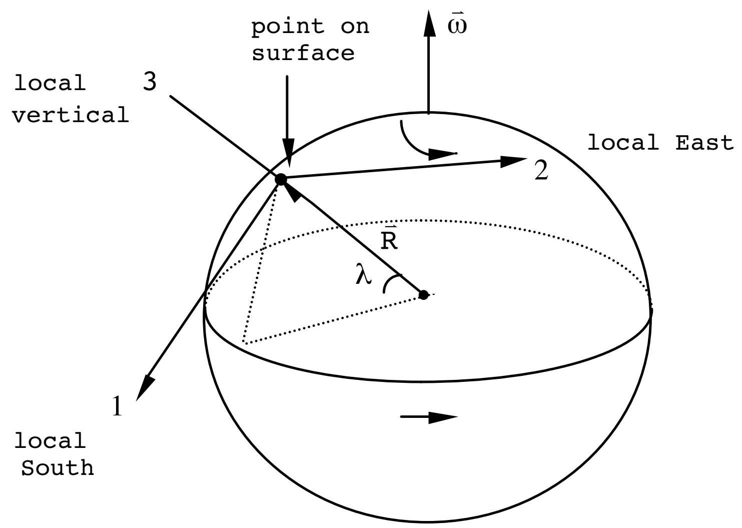
integrate $\Rightarrow r^2\omega = \frac{\ell}{m}$, $\omega = \frac{\ell}{mr^2}$, as before (conservation of angular momentum). "1" components:

$$m\ddot{r} = -\frac{dU}{dr} - m\left(-\omega^2 r\right),$$

$$\Rightarrow \ddot{r} - \omega^2 r = -\frac{1}{m}\frac{dU}{dr}.$$

or $\ddot{r} - \frac{\ell^2}{m^2 r^3} = -\frac{dU}{dr}$, also as before.

There are other choices of noninertial coordinate systems which can simply motion problems near the Earth's surface. Consider the choice:



For this choice we have

$$\vec{F}_{eff} = \vec{F} - m\left(\ddot{\vec{R}} + \overset{0}{\dot{\vec{\omega}}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \dot{\vec{r}}\right). \tag{10.44}$$

We need to compute $\ddot{\vec{R}}$. Using the general formula

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A},$$

for any $\vec{A}$, we get

$$\left(\dot{\vec{R}}\right)_f = \vec{\omega} \times \vec{R},$$

since $\left(\frac{d\vec{R}}{dt}\right)_r = 0$. Again applying the above general equation, we get

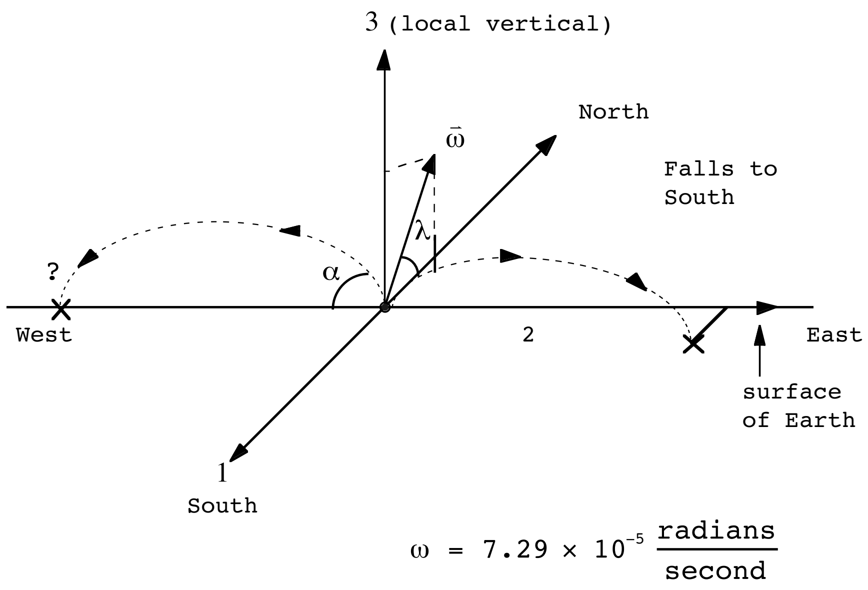
$$\left(\frac{d^2\vec{R}}{dt^2}\right)_f = \vec{\omega} \times \left(\frac{d\vec{R}}{dt}\right)_f = \vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right).$$

Notice that we now get

$$\vec{F}_{eff} = \vec{F} - m\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right) - \underset{\text{usually small}}{\underbrace{m\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)}} - 2m\vec{\omega} \times \dot{\vec{r}}. \tag{10.45}$$

10.4 DEFLECTION OF PROJECTILES NEAR EARTH'S SURFACE

Using this $\vec{F}_{eff}$, we can now investigate the motion of projectiles near the Earth's surface. Qualitatively:



$$\vec{\omega} = -\omega \cos\lambda\, \hat{e}_1 + \omega \sin\lambda\, \hat{e}_3 \,.$$

When projected initially to the East, the particle trajectory will gain a Southerly component due to 1. centrigulal force and 2. Coriolis force. However, if initially projected West, the force will now deflect the particle to the North. The interesting question is: which deflection will be bigger when projected to the West?

Take

$$\vec{F}_{\mathrm{eff}} \simeq \vec{F} - m\left(\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right) + 2\vec{\omega} \times \dot{\vec{r}}\right).$$

To $0^{\underline{th}}$ order, we have

$$\vec{R} = \underset{\substack{\uparrow \\ \text{radius of the Earth}}}{R_E}\hat{e}_3,$$

$$\Rightarrow \begin{cases} r_1(t) = 0, \\ r_2(t) = -\left(v_0 \cos\alpha\right) t \ \text{(to the West if } v_0 > 0), \\ r_3(t) = -\frac{1}{2} g t^2 + \left(v_0 \sin\alpha\right) t \ \text{(does not include effect of rotation)}. \end{cases}$$

Work out $-\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right)$ term:

$$\begin{aligned} \vec{\omega} \times \vec{R} &= \left(-\omega \cos\lambda \hat{e}_1 + \omega \sin\lambda \hat{e}_3\right) \times \left(R_E \hat{e}_3\right), \\ &= \omega R_E \cos\lambda \hat{e}_2, \end{aligned}$$

$$\begin{aligned} -\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right) &= -\left(-\omega \cos\lambda \hat{e}_1 + \omega \sin\lambda \hat{e}_3\right) \times \left(\omega R_E \cos\lambda \hat{e}_2\right), \\ &= \omega^2 R_E \cos^2\lambda \hat{e}_3 + \underset{\substack{\uparrow \\ \text{term we are interested in}}}{\omega^2 R_E \sin\lambda \cos\lambda \hat{e}_1}. \end{aligned}$$

Work out Coriolis term:

$$2\vec{\omega} \times \dot{\vec{r}} \simeq -2\left(-\omega \cos\lambda \hat{e}_1 + \omega \sin\lambda \hat{e}_3\right) \times \left(-v_0 \cos\alpha \hat{e}_2 \left(-gt + v_0 \sin\alpha\right)\hat{e}_3\right)$$

$$\simeq -2\omega \cos\lambda \, v_0 \cos\alpha \, \hat{e}_3 \; -2\omega \cos\lambda \left(-gt + v_0 \sin\alpha\right)\hat{e}_2$$

$$-2\omega \sin\lambda \, v_0 \cos\alpha \, \hat{e}_1$$

$$\uparrow$$

This is the term
we are interested in

Plugging these results back into our $\vec{F}_{eff}$ equation, we find

$$\left(\vec{F}_{eff}\right)_1 = m\ddot{r}_1 \simeq \omega^2 R_E \sin\lambda \cos\lambda - 2\omega \sin\lambda \, v_0 \cos\alpha \, .$$

If our initial condition is that $r_1(t) = 0$ at t = 0, then

$$r_1(t) \simeq \frac{1}{2}\left(\omega^2 R_E \sin\lambda \cos\lambda - 2\omega \sin\lambda \, v_0 \cos\alpha\right)t^2 \, .$$

Now eliminate the time, t, by using the 0th order equation,

$$r_3(t) = 0 = -\frac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t \, ,$$

$$\Rightarrow t \simeq \frac{2v_0 \sin\alpha}{g} \, .$$

This is the approximate time it takes for the projectile to hit the ground. Therefore

$$r_1(t) \simeq \frac{1}{2}\left(\omega^2 R_E \sin\lambda \cos\lambda - 2\omega \sin\lambda \, v_0 \cos\alpha\right)\frac{4v_0^2 \sin^2\alpha}{g^2} \, .$$

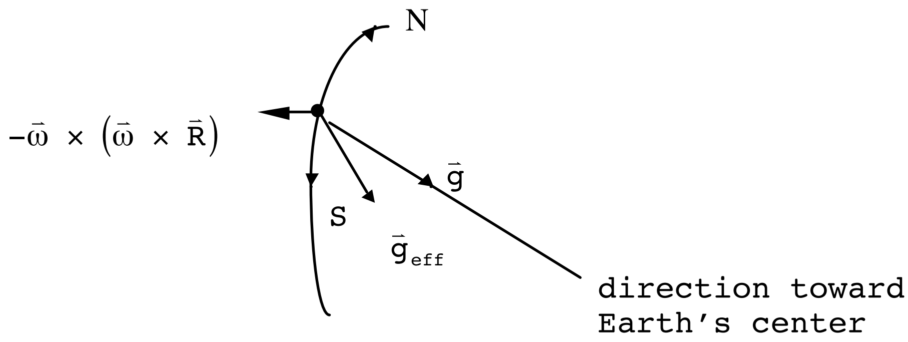
Question: which effect is larger? Depends on initial velocity, v_o. “Break even” velocity is

$$\left(v_0\right)_{BE} = \frac{\omega R_E \cos\lambda}{2\cos\alpha} \, . \qquad \left(\frac{\omega R_E}{2} \simeq 232 \, \frac{\text{meters}}{\text{sec}}\right)$$

Depends on latitude, cos α. (In this problem I have not been very careful about taking care of the Earth’s curvature. The above considerations only hold for short range projectiles.)

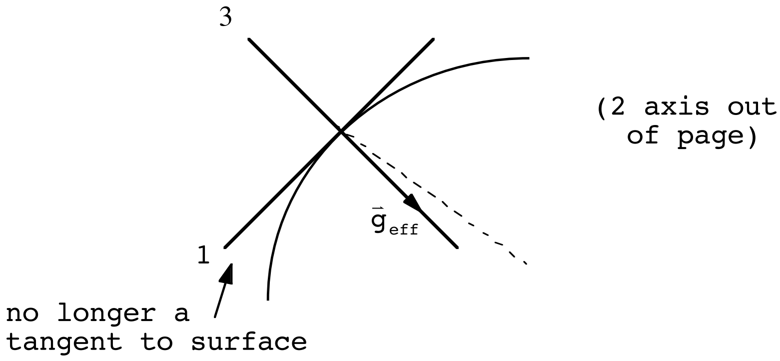
10.5 DEFLECTIONS FOR DROPPED OBJECTS

Let me now introduce another coordinate system which is useful for calculating the deflection of projectiles relative to the *gravitational vertical*. Remember:



$$\vec{g}_{eff} = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{R}).$$

Take new, skewed axes along $\vec{g}_{eff}$:



Then since

$$\vec{F}_{eff} = \underbrace{\vec{F} - m\vec{\omega} \times (\vec{\omega} \times \vec{R})}_{m\vec{g}_{eff}} - m\left(\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \dot{\vec{r}}\right),$$

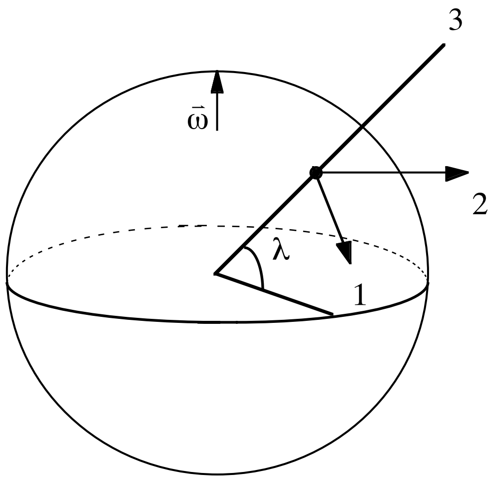
$$\vec{F}_{eff} = m\vec{a}_{eff},$$

$$\Rightarrow \vec{a}_{eff} = \vec{g}_{eff} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \dot{\vec{r}}. \tag{10.46}$$

where now $\vec{g}_{eff} = g_{eff}\,\hat{e}_3$ only. This is useful in discussing the deflection of particles relative to the local gravitational vertical, which can be established with a plum-bob, say. For example, if we had done the projectile problem above in the "skewed" frame, the term proportional to R_E in the form for $r_1(t)$ would have been absent. Then, to first order in ω:

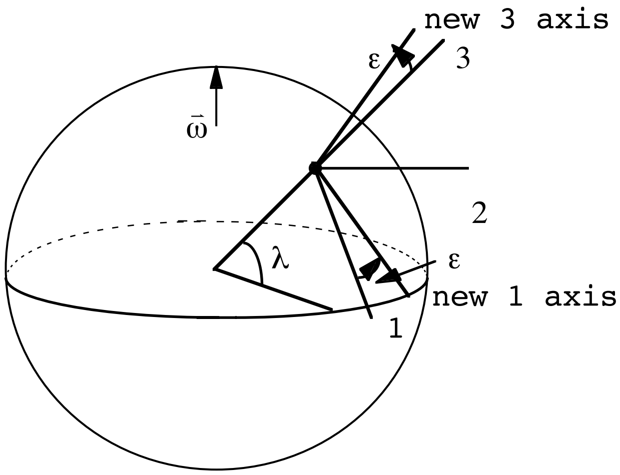
$$\vec{a}_{eff} \simeq -g_{eff}\,\hat{e}_3 - 2\vec{\omega} \times \dot{\vec{r}}. \tag{10.47}$$

In this new frame, we need to find $\vec{\omega}$. We had



$$\begin{cases} \omega_1 = -\omega\cos\lambda \\ \omega_2 = 0 \\ \omega_3 = \omega\sin\lambda \end{cases}$$

After we "skew" it by an angle, it looks like (rotation is around 2 axis):



Clearly, we have

$$\begin{cases} \omega_1 = -\omega \cos(\lambda + \varepsilon), \\ \omega_2 = 0, \\ \omega_3 = \omega \sin(\lambda + \varepsilon). \end{cases}$$

However, ε is a small angle (smaller than .002 radians) and will be neglected below. Now imagine dropping an object from a height, h.

In 0th order:

$$\ddot{r}_3 = -g_{eff}, \qquad \ddot{r}_{1,2} = 0,$$

$$\Rightarrow \dot{\vec{r}} = -g_{eff} t \hat{e}_3 \;, \; \vec{r} = -\frac{1}{2} g_{eff} t^2 \hat{e}_3 \; \left(\text{if } \vec{r} = 0 \text{ at } t = 0\right)$$

1st order (in ω) correction:

$$\left.\begin{aligned} \omega_1 &\simeq -\omega \cos\lambda, \\ \omega_2 &= 0, \\ \omega_3 &\simeq \omega \sin\lambda. \end{aligned}\right\} \text{as explained above}$$

Then,

$$\begin{aligned} -2\vec{\omega} \times \dot{\vec{r}} &\simeq 2\left(-\omega \cos\lambda \hat{e}_1 + \omega \sin\lambda \hat{e}_3\right) \times \left(-g_{eff}\, t \hat{e}_3\right), \\ &= 2\omega \cos\lambda g_{eff} + \hat{e}_2 \,. \end{aligned}$$

We now get

$$\ddot{r}_3 = -g_{eff}\,, \qquad \ddot{r}_1 = 0, \quad \ddot{r}_2 \simeq 2\,\omega g_{eff}\, t \cos\lambda\,.$$

Integrating twice on $\ddot{r}_2$, we get

$$r_2(t) \simeq \frac{1}{3}\,\omega g_{eff}\, t^3 \cos\lambda\,.$$

Of course, we have

$$t^2 \simeq \frac{2h}{g_{eff}},$$

$$\Rightarrow r_2(t) \simeq \frac{1}{3}\,\omega \left(\frac{8h^3}{g_{eff}}\right)^{1/2} \cos\lambda\,.$$

Since $r_2 > 0$, the deflection is to the East. You will study this problem further (to second order in ω^2) in a further HW problem.

10.6 FOCAULT PENDULUM

Last problem: the Foucault pendulum. Again, use our "skewed" coordinate system. Only changes:

$$g \rightarrow \vec{F}_{eff} = \underbrace{\vec{F}}_{\text{new external force}} + m\vec{g}_{eff} - m\left[\vec{\omega} \times \underbrace{\left(\vec{\omega} \times \vec{r}\right)}_{\text{small (ignore)}} + 2\vec{\omega} \times \dot{\vec{r}}\right] g_{eff}$$

Say

$$\vec{F} = F_1\,\hat{e}_1 + F_2\,\hat{e}_2 + F_3\,\hat{e}_3\,,$$

$$\vec{g}_{eff} = -g_{eff}\,\hat{e}_3\,,$$

$$\vec{\omega} = -\omega\cos\lambda\,\hat{e}_1 + \omega\sin\lambda\,\hat{e}_3\,.$$

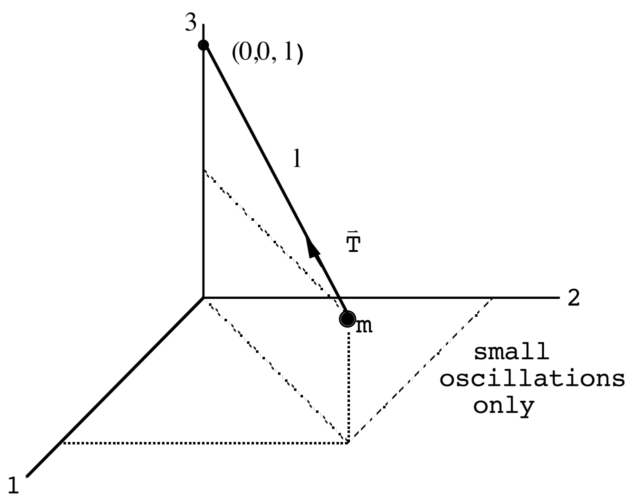
Put it all together:

$$m\ddot{r}_1 = F_1 + 2m\omega\dot{r}_2\sin\lambda\,,$$

$$m\ddot{r}_2 = F_2 - 2m\omega\left(\dot{r}_2\sin\lambda + \dot{r}_3\cos\lambda\right),$$

$$m\ddot{r}_3 = F_3 - mg_{eff} + 2m\omega\dot{r}_2\cos\lambda\,.$$

Consider:



$$F_1 = -\frac{r_1}{\ell}T,\quad F_2 = -\frac{r_2}{\ell}T,\quad F_3 = \frac{\left(\ell - r_3\right)}{\ell}T.$$

Neglect $\ddot{r}_3$ and $\frac{r_3}{\ell} \approx 0$. 3rd eqn above becomes

$$\Rightarrow \; T \simeq mg_{eff} - \underbrace{2m\omega\dot{r}_2 \cos\lambda}_{\text{small but} \neq 0}.$$

(not a constant)

Substitute this value of T above:

$$m\ddot{r}_1 = \frac{-r_1}{\ell}\underset{\substack{\uparrow \\ \text{small}}}{\left[2m\omega\dot{r}_2 \cos\lambda\right]} + 2m\omega\dot{r}_2 \sin\lambda,$$

$$m\ddot{r}_2 = \frac{-r_2}{\ell}\underset{\substack{\uparrow \\ \text{small}}}{\left[2m\omega\dot{r}_2 \cos\lambda\right]} - 2m\omega\left[\dot{r}_1 \sin\lambda + \underset{\substack{\uparrow \\ \text{small}}}{\dot{r}_3 \cos\lambda}\right]$$

Set $\alpha^2 \equiv \dfrac{g_{eff}}{\ell}$. Then

$$\ddot{r}_1 + \alpha^2 r_1 = 2\omega \dot{r}_2 \sin\lambda ,$$

$$\ddot{r}_2 + \alpha^2 r_2 = -2\omega \dot{r}_1 \sin\lambda .$$

There is a dynamic coupling between the motions. Can solve both at once using complex numbers:

$$\begin{aligned}\left(\ddot{r}_1 + i\ddot{r}_2\right) + \alpha^2\left(r_1 + ir_2\right) &= 2\omega\left(\dot{r}_2 - i\dot{r}_1\right)\sin\lambda ,\\ &= -2\omega i\left(\dot{r}_1 + i\dot{r}_2\right)\sin\lambda ,\end{aligned}$$

or

$$\dot{q} + \alpha^2 q = -2\omega i \dot{q} \sin\lambda .$$

where $q \equiv r_1 + ir_2$. This is solved by assuming

$$q = A\, e^{\lambda t} ,$$

where A, λ are to be determined. Substitute above:

$$\Rightarrow \lambda^2 + 2i\omega\lambda \sin\lambda + \alpha^2 = 0 ,$$

$$\Rightarrow \lambda = \frac{1}{2}\left[-2i\omega \sin\lambda \pm \sqrt{-4\omega^2 \sin^2\lambda - 4\alpha^2}\right],$$

$$\lambda = -i\omega \sin\lambda \pm i\sqrt{\omega^2 \sin^2\lambda + \alpha^2} .$$

But $\omega^2 \sin^2\lambda << \alpha^2 \left(= \dfrac{g}{\ell}\right)$ for the Earth, so

$$\lambda \simeq -i\omega \sin\lambda \pm i\alpha .$$

General solution: (A,B real)

$$g(t) = \left\{Ae^{i\alpha t} + Be^{-i\alpha t}\right\} e^{-i\omega t \sin\lambda} .$$

A,B are fixed by initial conditions. Write it out:

$$A\left(\cos\alpha t + i\sin\alpha t\right) + B\left(\cos\alpha t - i\sin\alpha t\right) = C_1\cos\alpha t + iC_2\sin\alpha t,$$

$$\text{Let} \quad C_1 = A + B\,,\; C_2 = A - B\,,$$

$$\Rightarrow q(t) = \left(C_1\cos\alpha t + iC_2\sin\alpha t\right)e^{-i\omega t\sin\lambda}\,.$$

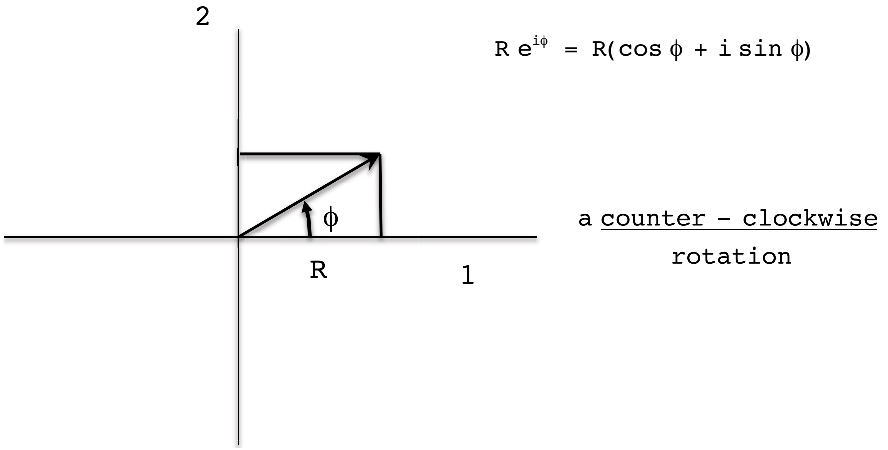
There are 2 parts to the motion with different frequencies since $\omega << \alpha$. Let's say $e^{-i\omega t\sin\lambda} = 1$. Then, since

$$r_1 = \text{Re}\,q(t)\,, \qquad r_2 = \text{Im}\,q(t)$$

$$= C_1\cos\alpha t \qquad = C_2\cos\alpha t$$

$$\Rightarrow \frac{r_1^2}{C_1^2} + \frac{r_2^2}{C_2^2} = 1\,.$$

Eq$^{\underline{n}}$ of an ellipse. However, we made r_1 and r_2 coordinates in the complex plane. The factor $e^{-i\omega t\sin\lambda}$ is just a *rotation* in the complex plane:

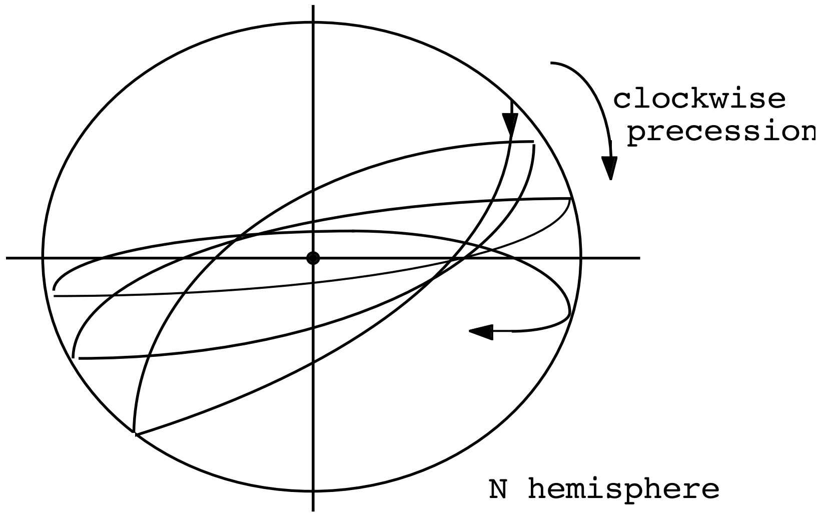


The real and imaginary parts of the complex number g(t) are just the r_1,r_2 components of the real motion. Therefore, a rotation in the complex plane is also describing a rotation of the vector $r = r_1\,\hat{e}_1 + r_2\,\hat{e}_2$ in real space. Because of the minus sign in $e^{-i\omega t\sin\lambda}$, this is a *clockwise* rotation in coordinate space. (It would be counter-clockwise in the Southern hemisphere.) Time it takes to complete a rotation:

$$2\pi = \omega T\sin\lambda \Rightarrow T = \frac{2\pi}{\omega\sin\lambda}\,,$$

$$\omega = \frac{2\pi}{1\text{ day}} \Rightarrow T = \frac{1}{\sin\lambda}\text{ days}\,.$$

Goes around once a day at $l = 90^0$ (N or S poles) and does not precess at all at the equator ($l = 0^0$). Actual motion looks like:



Looks like the precession of the orbit of a planet under general relativity, but the forces here certainly are not central. The "force" that makes it precess, in fact, is purely fictional.

10.7 PROBLEMS

1. (a) I showed in the notes that instantaneously

$$\vec{r}\,' = \vec{r} + \vec{R}\,,$$

$$\dot{\vec{r}}\,' = \dot{\vec{r}} + \dot{\vec{R}} + \vec{\omega} \times \vec{r}\,,$$

$$\ddot{\vec{r}}\,' = \ddot{\vec{r}} + \ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{r} + 2\,\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\,.$$

Find the relation between $\dddot{\vec{r}}\,' = \dot{\vec{a}}\,'$ and $\dddot{\vec{r}} = \dot{\vec{a}}$. ($\dddot{\vec{R}} = \dot{\vec{A}}$); assume all derivatives of $\vec{\omega}$ to be zero.) Ans:

$$\dot{\vec{a}}\,' = \dot{\vec{a}} + \dot{\vec{A}} + 2\,\vec{\omega} \times \vec{a} + 3\,\vec{\omega} \times (\vec{\omega} \times \dot{\vec{r}}) + \vec{\omega} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}))\,.$$

(b) Continue this process and find the relation between $\ddot{\vec{a}}\,'$ and $\ddot{\vec{a}}$.

2. Show that the angular deviation ε of a plumb line from the true vertical at a point on the Earth's surface at a latitude λ is

$$\varepsilon \approx \frac{r_0\omega^2 \sin\lambda \cos\lambda}{g - r_0\omega^2 \cos^2\lambda}\,,$$

where r_0 is the Earth's radius and g is acceleration due to gravity.

3. By balancing centrifical "force" and gravitational force, find the orbital velocity of an object in a circular orbit just above the Moon's surface.

($R_{moon} = 1.74 \times 10^8$ cm, $M_{moon} = 7.35 \times 10^{25}$ gm,

$G = 6.67 \times 10^{-8}$ cm $\dfrac{\text{dyne} \times \text{cm}^2}{\text{gm}^2}$)

↑ Newton's gravitational constant

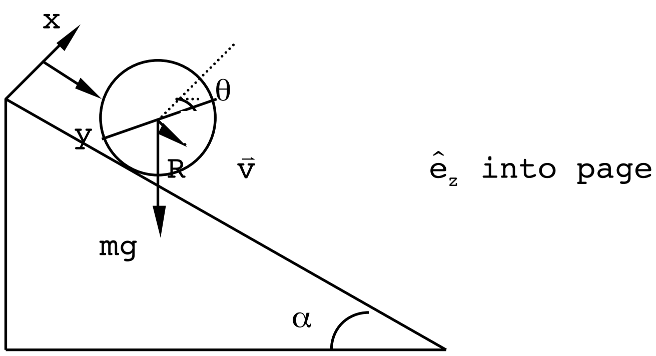
4. At latitude, $\lambda = 31.5°$, how many hours does it take for the plane of a Focault pendulum to complete a revolution?

11 RIGID BODY MOTION

11.1 CONCEPT OF A RIGID BODY

Attention up to now has been focused on individual point particles or collections of point particles for the most part. Not very realistic.

Consider the example:



Now have to deal with the disk, not as a collection of individual particles, but as a whole, i.e., a *rigid* body. For the disk,

$$I\ddot{\theta} = N_z \quad , \quad I = \frac{1}{2} mR^2$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \underline{\text{like}}\text{: } ma_z = F_z & \underline{\text{like}}\text{: mass} \end{array}$$

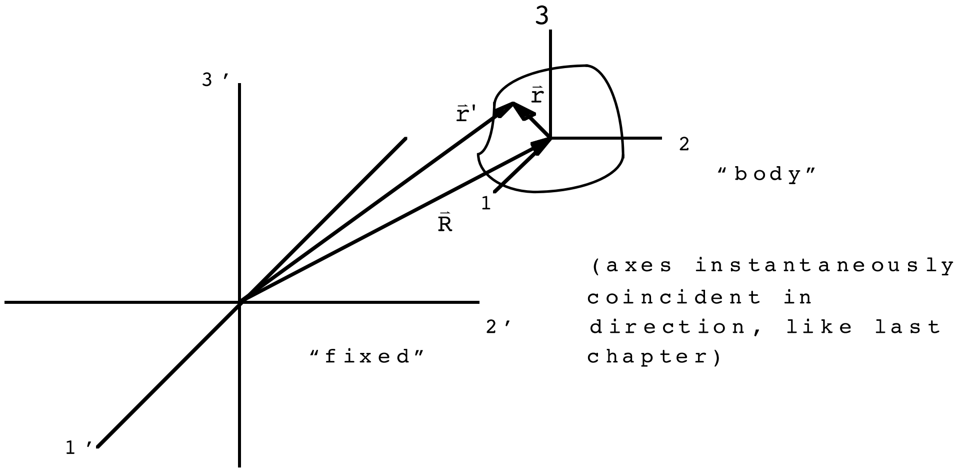
"I" is a way of representing a rigid body in a simple manner. Kinetic energy is,

$$T = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} I\dot{\theta}^2,$$

separated into rotational and translational parts. First, let's understand this result for kinetic energy and then derive the form for "I".

11.2 INSTANTANEOUS KINETIC ENERGY IN BODY FRAME

Picture:



The $\vec{r}$ axes are considered *fixed* to the body.

Connection from last Chapter:

$$\underset{\substack{\uparrow \\ \text{measured} \\ \text{in ``fixed''} \\ \text{frame}}}{\dot{\vec{r}}'} = \underset{\substack{\uparrow \\ \text{components measured} \\ \text{in body frame}}}{\dot{\vec{r}}} + \dot{\vec{R}} + \vec{\omega} \times \vec{r}\,. \tag{11.1}$$

We are going to be describing the motion from the body frame. (Why not just describe it from the "fixed" axes? Wait a sec.) Clearly, we have

$$\dot{\vec{r}} = 0, \qquad \left(\ddot{\vec{r}} = 0, \dddot{\vec{r}} = 0, \ldots\right) \tag{11.2}$$

since we are keeping every point in the body fixed relative to body axis. In order to be able to deal with the motion of rigid bodies it is crucial that both the kinetic and potential energies separate into center of mass and relative motion pieces. We have already seen this is true for a collection of point particles or planets interacting via forces derived from potentials. Relative to the fixed frame we have

$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha^{\,\prime}\,, \text{ (usually use } \vec{r}_\alpha \text{ for fixed frame)} \tag{11.3}$$

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,\prime\,2}\,, \tag{11.4}$$

Using Eq.(11.1), we have

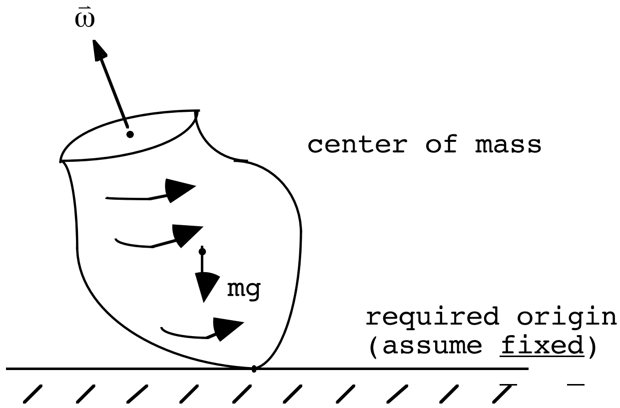
$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \dot{\vec{R}}\cdot\sum_\alpha m_\alpha\left(\vec{\omega}\times\vec{r}_\alpha\right) + \frac{1}{2}\sum_\alpha m_\alpha\left(\vec{\omega}\times\vec{r}_\alpha\right)^2.$$

But if we choose the origin of the body system to be at the center of mass, we have

$$\sum_\alpha m_\alpha \vec{r}_\alpha = 0\,,$$

$$\Rightarrow \quad T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha\left(\vec{\omega}\times\vec{r}_\alpha\right)^2. \text{ (separated)} \tag{11.5}$$

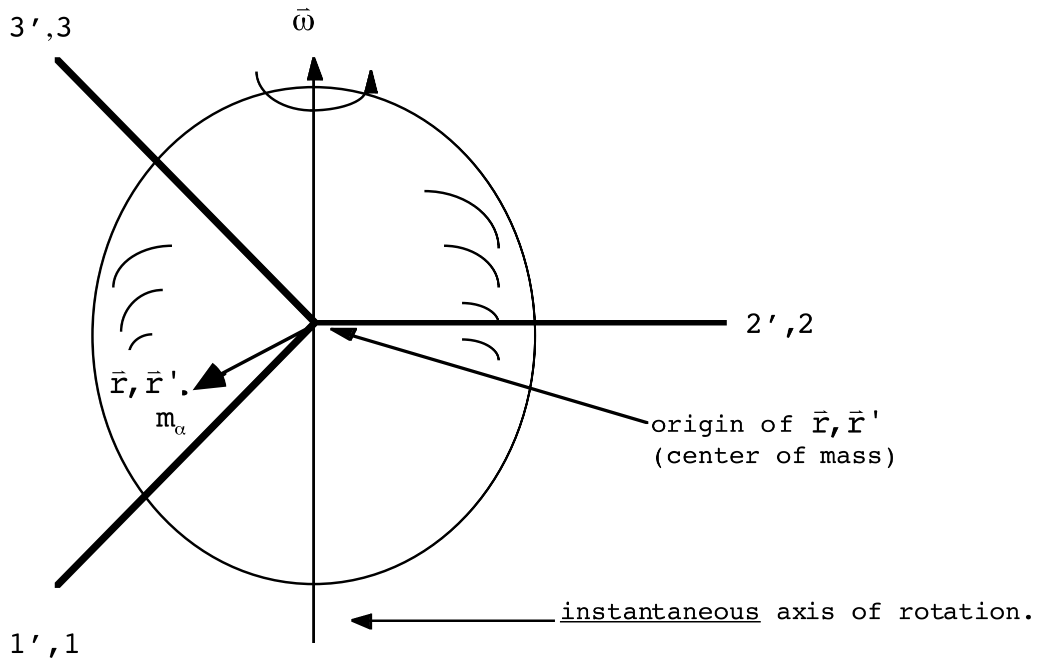
Actually, there is another possibility for describing rigid body motion: if $\dot{\vec{R}} = 0$ for some point other than the center of mass. This means the origins of the fixed and body systems are not in relative motion. We can then, with no loss in generality, assume the two origins coincide. It is important to realize what $\dot{\vec{R}} = 0$ means. Since the $\vec{r}$ axes are fixed in the body, this is a requirement that at least one point associated with the body is not moving in some inertial frame. And, if we want this description to be valid over finite time intervals, that point must not move (equivalent to $\ddot{\vec{R}} = 0, \dddot{\vec{R}} = 0, \ldots$). For example, one can describe in this case a top with one point fixed:



Can not describe a situation where the tip of the top is moved around arbitrarily, however.

11.3 ANGULAR MOMENTUM AND THE INERTIA TENSOR

Axes situation for an arbitrary spinning object:



Point: instantaneous axis of rotation passes through the center of mass if there is no external torque. An instantaneous axis of rotation always exists for a rigid body. Sometimes given, sometimes it must be deduced. We will learn later that even if $\dot{\vec{L}} = 0$, this does not imply $\dot{\vec{\omega}} = 0$ for a rigid body in general.

We know that

$$\dot{\vec{r}}'_\alpha = \vec{\omega} \times \vec{r}_\alpha . \tag{11.6}$$

The angular momentum associated with this mass element is

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}'_\alpha = m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}'_\alpha = m_\alpha \left[\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)\right]. \tag{11.7}$$

Sum on a and use a vector identity,

$$\vec{L} \equiv \sum_\alpha \vec{L}_\alpha = \sum_\alpha m_\alpha \left[r_\alpha^2 \vec{\omega} - (\vec{r}_\alpha \cdot \vec{\omega})\, \vec{r}_\alpha\right]. \tag{11.8}$$

Write this out explicily in terms of components. Useful notation (using "x" instead of "r" from now on in this chapter):

$$\underset{\substack{\uparrow \\ \text{particle label}}}{x_{\alpha i}} \leftarrow \text{component label}$$

Then

$$L_i = \sum_\alpha m_\alpha \left[\omega_i \sum_k x_{\alpha k}^2 - x_{\alpha i} \sum_j x_{\alpha j}\, \omega_j\right],$$

$$= \sum_\alpha m_\alpha \sum_j \left[\omega_j\, \delta_{ij} \sum_k x_{\alpha k}^2 - \omega_j\, x_{\alpha i}\, x_{\alpha j}\right],$$

$$= \sum_j \omega_j \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k x_{\alpha k}^2 - x_{\alpha i}\, x_{\alpha j}\right]. \tag{11.9}$$

Define ("Inertia tensor"; more on this later)

$$I_{ij} \equiv \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k x_{\alpha k}^2 - x_{\alpha i}\, x_{\alpha j}\right]. \tag{11.10}$$

Then, in component notation (notice $I_{ij} = I_{ji}$),

$$L_i = \sum_j I_{ij}\, \omega_j\,, \tag{11.11}$$

or in matrix notation,

$$\underset{\substack{\uparrow \\ \text{column} \\ \text{matrix}}}{L} = \underset{\substack{\uparrow \\ \text{3X3 square matrix}}}{I}\, w \leftarrow \text{column matrix} \tag{11.12}$$

Terminology:

I_{ii}: "moments of inertia"

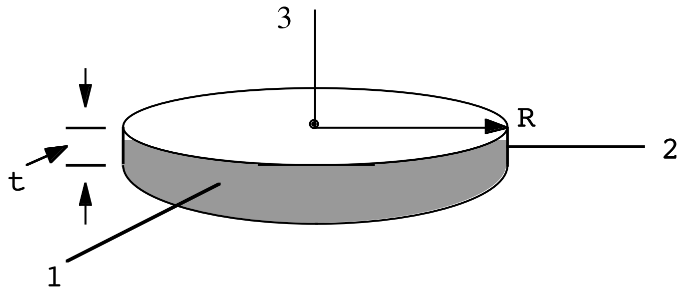
I_{ij} (i≠j): "products of inertia"

Because I_{ij} for $i \neq j$ is nonzero in general, can now understand the above comment that $\dot{\vec{L}} = 0$ does not in general imply. Of course, in the usual case that one considers the object to be a continuous distribution of mass rather than a collection of discrete elements, one has

$$\sum_{\alpha} \underset{\substack{\uparrow \\ \text{(volume element)}}}{m_\alpha} \rightarrow \int_V dv\, \rho(\vec{r})\ , \text{ so that}$$

$$I_{ij} = \int_v dv\, \rho\left(\vec{r}\right)\left[\delta_{ij} \sum_k x_k^2 - x_i x_j\right]. \tag{11.13}$$

Do a cylinder as an example: (density = ρ = constant. I usually use ρ, z, ϕ as the cylindrical coordinates, but because ρ is being used as density, I will use the set r, z, ϕ instead.)



$$I_{33} = \int \underbrace{r dr d\phi dz}_{\substack{dv \text{ in} \\ \text{cylindrical} \\ \text{coord.}}} \rho \left[\underbrace{\left(x^2 + y^2 + z^2\right) - z^2}_{x^2+y^2=r^2}\right],$$

$$\Rightarrow I_{33} = \frac{1}{4} R^4 2\pi t \rho. \text{ But } \rho = \frac{m}{\pi R^2 t},$$

$$\Rightarrow I_{33} = \frac{1}{2} mR^2.$$

What about around other axes?

$$I_{11} = \rho \int r\,dr\,d\phi\,dz\left[\left(x^2 + y^2 + z^2\right) - x^2\right],$$

$$(x = r\cos\phi,\ y = r\sin\phi)$$

$$I_{11} = \rho \int dr\,d\phi\,dz\left[\overset{(1)}{r^3 \sin^2\phi} + \overset{(2)}{rz^2}\right],$$

$$(1) \quad = \rho t \int dr\,d\phi\, r^3 \sin^2\phi = \rho t\,\frac{1}{4} R^4 \left(\frac{\phi}{2} - \frac{1}{4}\sin 2\phi\right)\Bigg|_0^{2\pi},$$

$$= \frac{\pi}{4}\rho t R^4.$$

$$② = 2\pi\rho \int dr\, dz\, rz^2 = 2\pi\rho \frac{1}{2} R^2 \int_{-t/2}^{t/2} dz\, z^2 ,$$

$$= \frac{\pi}{12} \rho t^3 R^2$$

$$\Rightarrow I_{11} = ① + ② = \rho \left(\frac{\pi}{4} tR^4 + \frac{\pi}{12} R^2 t^3 \right)$$

$$= m \left[\frac{1}{4} R^2 + \frac{1}{12} t^2 \right].$$

Similarly for I_{22}. In addition, one can see that the I_{ij}, $i \neq j$, vanish by symmetry. Example:

$$I_{12} = -\int dv\, \rho(\bar{r}) \underbrace{xy}_{r^2 \sin\phi \cos\phi} = 0 .$$

So, in this case we have

$$\bar{L} = I_{33}\omega_3 \hat{e}_3 + I_{11}\omega_1 \hat{e}_1 + I_{22}\omega_2 \hat{e}_2 .$$

Since $I_{11} \neq I_{33}$, can see that $\bar{L}$ and $\bar{\omega}$ are pointed in different directions. Axes for which the I_{ij}, $i \neq j$, vanish are special and are called *principal axes*. We will find out how to identify them in a little bit.

11.4 TRANSFORMATION PROPERTIES OF THE INERTIA TENSOR

It is now clear why we are describing motion from the body axes. It is clear that the I_{ij} take on different values for different orientations of our axes. This is why in general we do not try to describe the motion from the fixed frame. As the body moves, the I_{ij} would become functions of time. (Sometimes it is convenient to describe motion in the fixed frame, however, if the I_{ij} are constant there. This is the case for a disk rolling down an inclined plane, say.)

Can relate the kinetic energy of rotation to I_{ij}. We had

$$T_{rot} = \frac{1}{2} \sum_{\alpha} m_\alpha \left(\bar{\omega} \times \bar{r}_\alpha \right)^2 ,$$

$$\Rightarrow T_{rot} = \frac{1}{2} \sum_{\alpha} m_\alpha \left[\bar{\omega}^2 \bar{r}_\alpha^{\,2} - \left(\bar{\omega} \cdot \bar{r}_\alpha \right)^2 \right].$$

or

$$
\begin{aligned}
T &= \frac{1}{2}\sum_{\alpha} m_\alpha \left[\sum_i \omega_i^2 \sum_k x_{\alpha k}^2 - \sum_i \omega_i x_{\alpha i} \sum_j \omega_j x_{\alpha j}\right], \\
&= \frac{1}{2}\sum_{\alpha}\sum_{i,j} m_\alpha \left[\omega_i \omega_j \delta_{ij} \sum_k x_{\alpha k}^2 - \omega_i \omega_j x_{\alpha i} x_{\alpha j}\right], \\
&= \frac{1}{2}\sum_{i,j} \omega_i \omega_j \sum_{\alpha} m_\alpha \left[\delta_{ij} \sum_k x_{\alpha k}^2 - x_{\alpha i} x_{\alpha j}\right], \\
\Rightarrow T_{rot} &= \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = \frac{1}{2}\sum_i L_i \omega_i .
\end{aligned}
\tag{11.14}
$$

We have

$$T' = \frac{1}{2}\sum_{i,j} I'_{ij}\omega'_i\omega'_j .$$

In a rotated coordinate system:

$$T = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j .$$

T is a scalar, ω_i are vectors under rotations:

$$T' = T.$$

$\omega_k = \sum_{\ell} \lambda_{k\ell}\omega'_\ell$ (actually, is ω'_k a *pseudovector*:

$\omega_k = (\det \lambda)\sum_{\ell} \lambda_{k\ell}\omega'_\ell$)

so then

$$T = \frac{1}{2}\sum_{\substack{i,j\\k,\ell}} I_{ij}\lambda_{ik}\lambda_{j\ell}\omega'_k\omega'_\ell = T'.$$

In order for this equation to reproduce $T' = \frac{1}{2}\sum_{k,\ell} I'_{k\ell}\omega'_k\omega'_\ell$, we must have that

$$I'_{k\ell} = \sum_{i,j} I_{ij}\lambda_{ik}\lambda_{j\ell}, \text{ a } \textit{tensor}.$$

Can write this as a matrix equation also:

$$I'_{k\ell} = \sum_{i,j} \lambda^T_{ki} I_{ij}\lambda_{j\ell},$$

or

$$I' = \lambda^T I \lambda \quad (\lambda, I \text{ are } 3 \times 3 \text{ matrices})$$

Remember that $\lambda^T = \lambda^{-1}$, so this relation may be used to get I in terms of I':

$$\lambda\left(\lambda^T I \lambda\right)\lambda^T = \lambda I' \lambda^T,$$

or (matrix notation)

$$\mathtt{I} = \lambda \mathtt{I}' \lambda^{T} \qquad \text{(“similarity transformation”)} \tag{11.15}$$

or (index notation)

$$\mathtt{I}_{k\ell} = \sum_{i,j} \lambda_{ki} \mathtt{I}'_{ij} \lambda^{T}_{j\ell} = \sum_{ij} \mathtt{I}'_{ij} \lambda_{ki} \lambda_{\ell j} \,. \tag{11.16}$$

11.5 PRINCIPAL AXES

How do we find the axes for which I_{ij}, $i \neq j$, vanish, and do such axes *always* exist?

Let us assume that $\mathtt{I}'_{ij}$ is diagonal:

$$\mathtt{I}'_{ij} = \mathtt{I}'_{i} \delta_{ij} \,. \qquad \text{(no sum on i)}$$

As a matrix:

$$\mathtt{I}' = \begin{pmatrix} \mathtt{I}'_1 & 0 & 0 \\ 0 & \mathtt{I}'_2 & 0 \\ 0 & 0 & \mathtt{I}'_3 \end{pmatrix} .$$

We have (switching the meaning of primed, unprimed quantities relative to (11.16))

$$\mathtt{I}'_{ij} = \sum_{k,\ell} \lambda_{ik} \lambda_{j\ell} \mathtt{I}_{k\ell} \,.$$

The left hand side, which is diagonal, may be written as $\delta_{ij} \mathtt{I}'_i$, where $\mathtt{I}'_i$ are the principal values of the inertia tensor. Multiply both sides by λ_{im} and sum on i:

$$\sum_{i} \mathtt{I}'_{i} \delta_{ij} \lambda_{im} = \sum_{k,\ell} \left(\sum_{i} \lambda_{im} \lambda_{ik} \right) \lambda_{j\ell} \mathtt{I}_{k\ell} \,,$$

$$\Rightarrow \mathtt{I}'_{j} \lambda_{jm} = \sum_{\ell} \lambda_{j\ell} \mathtt{I}_{m\ell} \,. \quad \texttt{(no j sum)}$$

Can write ℓhs as

$$\mathtt{I}'_{j} \lambda_{jm} = \sum_{\ell} \lambda_{j\ell} \mathtt{I}'_{j} \delta_{m\ell} \,.$$

Can now group these terms together as

$$\sum_{\ell}\left(I_{m\ell} - I'_j\delta_{m\ell}\right)\lambda_{j\ell} = 0\,. \qquad (11.17)$$

Free indices: j,m. This represents a set of 9 equations in 9 unknowns. That is, given a set of $I_{m\ell}$ around some given axes, we have derived a set of equations that set conditions on the $\lambda_{j\ell}$ necessary to find principal axes. For these to have a nontrivial solution, we must have the determinant of the coefficients of the $\lambda_{j\ell}$ vanish, for each j value (which is a free index). In matrix notation (the "secular equation"):

$$\det\begin{pmatrix} (I_{11} - I') & I_{12} & I_{13} \\ I_{21} & (I_{22} - I') & I_{23} \\ I_{31} & I_{32} & (I_{33} - I') \end{pmatrix} = 0\,,$$

or

$$\det\,(I - \underset{\substack{\uparrow \\ \text{unit matrix: } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}}{1}I') = 0.$$

This gives a cubic equation in I'. The 3 roots of this equation then give the values of the 3 principal moments of inertia:

$$I' = \begin{pmatrix} I'_1 & 0 & 0 \\ 0 & I'_2 & 0 \\ 0 & 0 & I'_3 \end{pmatrix}.$$

This is a completely general procedure and can be done for any given origin. This procedure gives the moments, but what axes are being referred to?

There is another way of viewing this procedure. General experession for $\vec{L}$:

$$L_i = \sum_j I_{ij}\omega_j\,,$$

or

$$\mathrm{L} = \mathrm{I}\omega. \qquad \text{(matrix language)}$$

What we are requiring above is equivalent to solving the equation

$$\underset{\substack{\uparrow \\ 3\times 3 \text{ matrix}}}{\mathrm{I}}\,\omega = \underset{\substack{\uparrow \\ \text{number ("eigenvalue")} \\ \text{(either } I_1', I_2', I_3' \text{ from above)}}}{\mathrm{I}'}\,\omega. \quad (\omega\text{: column matrix}) \tag{11.18}$$

Called *eigenvalue* problem. ω's which satisfly the above are called *eigenvectors*. General form:

$$\underset{\substack{\uparrow \\ \text{square matrix}}}{\mathtt{A}}\ \underset{\substack{\uparrow \\ \text{column matrix}}}{\mathtt{x}} = \overset{\substack{\text{number ("eigenvalue")} \\ \downarrow}}{\mathtt{A'}}\ \underset{\substack{\uparrow \\ \text{("eigenvector")}}}{\mathtt{x}}.$$

Values of *moments of inertia* about principal axes correspond to *eigenvalues*. *Directions of principal axes* correspond to *eigenvectors*. (One sees the same mathematics, called linear algebra, in many other places in physics, including quantum mechanics.)

Written in matrix language, the above is

$$(I - 1I')\omega = 0,$$

or in explicit component language (also follows from Eq.(11.17) by multiplying both sides by ω'_j, summing on j, and using $\omega_\ell = \sum_\ell \omega'_j \lambda_{j\ell}$),

$$\sum_j \left(I_{ij} - I'\delta_{ij}\right)\omega_j = 0. \quad (11.19)$$

Explicitly, 3 equations:

$$\left.\begin{aligned} \left(I_{11} - I'\right)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0, \\ I_{21}\omega_1 + \left(I_{22} - I'\right)\omega_2 + I_{23}\omega_3 &= 0, \\ I_{31}\omega_1 + I_{32}\omega_2 + \left(I_{33} - I'\right)\omega_3 &= 0. \end{aligned}\right\} \quad (11.20)$$

Again, the condition that there be a solution says det(I – I'1) = 0, as before. Get something new this way: the priciple axis eigenvectors. These are the linear combinations of ω_1, ω_2, ω_3 which are associated with each eigenvalue, $I'_{1,2,3}$. How do we find the eigenvectors? After we have determined the values of the eigenvalues, $I'_{1,2,3}$, we substitute one of the values back into the 3 above equations and solve for the ratio of values of ω_1: ω_2: ω_3. Why do we only get the ratio of values of $\omega_{1,2,3}$? If wi solves (i = 1,2,3; these are eigenvector labels, not vector components)

$$I\omega^i = I'_i\omega^i,$$

then $\tilde{\omega}^i = c\omega^i$, c arbitrary, also solves it. Thus, the normalization of the ω^i column vector is undetermined. This makes physical sense since we would not expect the magnitude of $\vec{\omega}_i$ (corresponding to the rate of rotation) to be determined, but simply it's direction (given by the ratio of components).

Notice that if $\vec{\omega}_i$ is given by (vector notation)

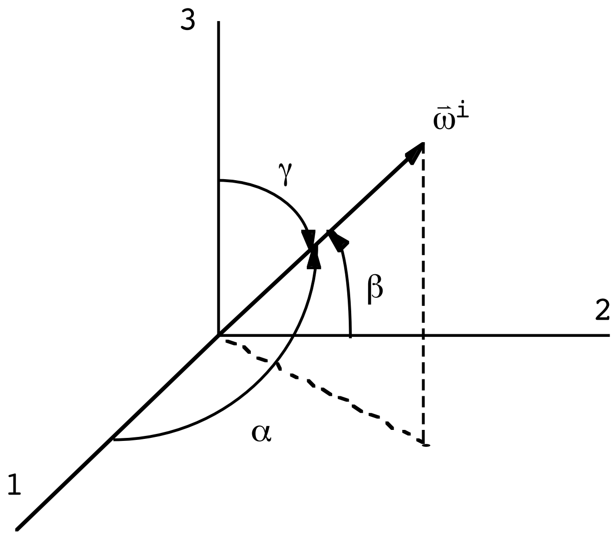
$$\vec{\omega}^i = \omega_1^i\hat{e}_1 + \omega_2^i\hat{e}_2 + \omega_3^i\hat{e}_3, \quad (11.21)$$

we can build a unit vector out of it by dividing by $\left|\vec{\omega}^{i}\right|$:

$$\hat{e}_{\omega^{i}} = \frac{\vec{\omega}^{i}}{\left|\vec{\omega}^{i}\right|} = \frac{\omega_1^{i}}{\omega^{i}}\,\hat{e}_1 + \frac{\omega_2^{i}}{\omega^{i}}\,\hat{e}_2 + \frac{\omega_3^{i}}{\omega^{i}}\,\hat{e}_3\,, \tag{11.22}$$

$$\omega^{i} \equiv \sqrt{\omega_1^{i2} + \omega_2^{i2} + \omega_3^{i2}}\,. \tag{11.23}$$

We then recognize these components as the direction cosines studied in Chapter 1:



$$\left.\begin{array}{l} \cos\gamma = \dfrac{\omega_3^{i}}{\omega^{i}} \\ \cos\beta = \dfrac{\omega_2^{i}}{\omega^{i}} \\ \cos\alpha = \dfrac{\omega_1^{i}}{\omega^{i}} \end{array}\right\} \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

So, if we know the $\omega_1^{i} : \omega_2^{i} : \omega_3^{i}$ ratio for some eigenvalue I', we can always find the *direction* of the corresponding principal axis. This represents a unique axis of rotation since the choice of origin fixes a point through which this vector points.

There is another important property of the eigenvectors, ω^i. Consider a situation where one has distinct (unequal) eigenvalues, $I_1' \neq I_2'$. Then (matrix notation)

$$\mathrm{I}\omega^1 = I_1'\omega^1 \;,\; \mathrm{I}\omega^2 = I_2'\omega^2 \;,$$

$$\Rightarrow \underset{\uparrow \atop \text{row matrix}}{\omega^2}\, \mathrm{I}\, \underset{\uparrow \atop \text{column matrix}}{\omega^1} = I_1'\omega^2\omega^1 \;,\; \omega^1\mathrm{I}\omega^2 = I_2'\omega^1\omega^2 \,.$$

But

$$\omega^2\mathrm{I}\omega^1 = \sum_{i,j} \omega_i^2 I_{ij}\omega_j^1 \overset{I_{ij} = I_{ji}}{=} \sum \omega_j^1 I_{ji}\omega_i^2 = \omega^1\mathrm{I}\omega^2 \;,$$

$$\omega^1\omega^2 = \sum_i \omega_i^1\omega_i^2 = \omega^2\omega^1 \;,$$

so

$$I_1'\omega^1\omega^2 = I_2'\omega^2\omega^1,$$

$$\Rightarrow \left(I_1' - I_2'\right)\omega^1\omega^2 = 0,$$

$$I_1' \neq I_2' \Rightarrow \omega^1\omega^2 = \vec{\omega}^1 \cdot \vec{\omega}^2 = 0.$$

Thus, eigenvectors of distinct eigenvalues are *orthogonal.* (point in ⊥ directions). If a double root of the secular equation occurs so that $I_1' = I_2'$, then nothing can be said about the relative directions of $\vec{\omega}^1$ and $\vec{\omega}^2$, other than that both are ⊥ to $\vec{\omega}^3$. However, there is no loss in generality if we *choose* $\vec{\omega}^1 \cdot \vec{\omega}^2 = 0$. One can prove that the principal axes can always be chosen to constitute an orthogonal set. Thus, for example we may always choose the eigenvalue orthogonalization/normalization condition, $\vec{\omega}^i \cdot \vec{\omega}^j = \delta_{ij}$.

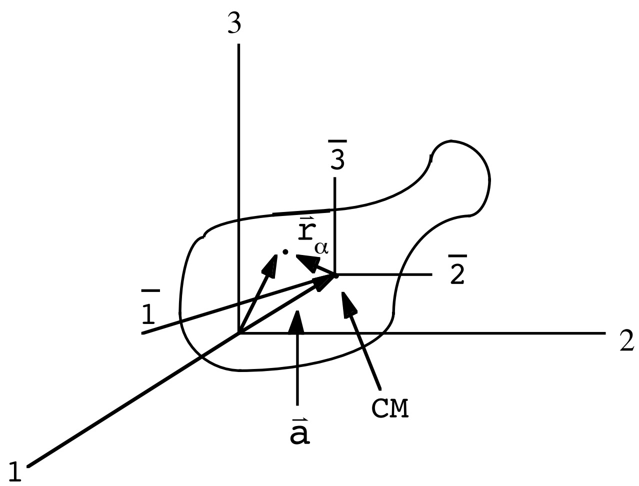
It is conceivable that the $I_{1,2,3}'$ could be *complex* since they correspond to the solution of a cubic equation. A slight variant of the above argument shows that the $I_{1,2,3}'$ are in fact always *real.*

11.6 PARALLEL AXIS THEOREM

Let's locate the CM of a body and calculate the inertia tensor about it and another set of axes *parallel* to the CM ones.

$1, 2, 3$ axes $: J_{ij}$

$\bar{1}, \bar{2}, \bar{3}$ axes $: I_{ij}$



On one hand, we have

$$\left(L\right)_i = \sum_\alpha \left(\vec{L}_\alpha\right)_i = \sum_\alpha m_\alpha \left(\vec{r}_\alpha \times \left(\vec{\omega} \times \vec{r}_\alpha\right)\right)_i ,$$

for the 1,2,3 axes. We already showed this may be written as

$$\left(\vec{L}\right)_i = \sum_j J_{ij}\omega_j ,$$

$$\Rightarrow \sum_\alpha m_\alpha \left(\vec{r}_\alpha \times \left(\vec{\omega} \times \vec{r}_\alpha\right)\right)_i = \sum_j J_{ij}\omega_j ,$$

where J_{ij} is the inertia tensor for the $1,2,3$ axes. Similarly, for the $\bar{1}, \bar{2}, \bar{3}$ axes we have

$$\sum_\alpha m_\alpha \left(\vec{\bar{r}}_\alpha \times \left(\omega \times \vec{\bar{r}}_\alpha\right)\right)_i = \sum_j I_{ij}\omega_j,$$

which defines the CM inertia tensor I_{ij}. Question: How are I_{ij} and J_{ij} related to one another? The connection is given once one recognizes that

$$\vec{r}_\alpha = \vec{a} + \vec{\bar{r}}_\alpha .$$

Then

$$\sum_\alpha m_\alpha \left(\vec{r}_\alpha \times \left(\vec{\omega} \times \vec{r}_\alpha\right)\right) = \overset{1}{\sum_\alpha m_\alpha \vec{\bar{r}}_\alpha \times \left(\vec{\omega} \times \vec{\bar{r}}_\alpha\right)} +$$

$$\overset{2}{\sum_\alpha m_\alpha \vec{a} \times \left(\vec{\omega} \times \vec{a}\right)} + \overset{3}{\sum_\alpha m_\alpha \left(\vec{a} \times \left(\vec{\omega} \times \vec{\bar{r}}_\alpha\right) + \vec{\bar{r}}_\alpha \times \left(\vec{\omega} \times \vec{a}\right)\right)} .$$

Lots of simplifications happen:

$$1 = \sum_j I_{ij}\omega_j , \quad (I_{ij}\text{: CM inertia tensor})$$

$$2 = M\,\vec{a} \times \left(\vec{\omega} \times \vec{a}\right) = M\left[\vec{\omega}a^2 - \vec{a}\left(\vec{\omega} \cdot \vec{a}\right)\right],$$

or (in index notation)

$$(②)_i = M \sum_j \left[\delta_{ij} a^2 - a_i a_j\right]\omega_j \,.$$

Also

$$③ = \bar{a} \times \Big(\bar{\omega} \times \underbrace{\Big(\sum_\alpha m_\alpha \bar{\bar{r}}_\alpha\Big)}_{0}\Big) + \underbrace{\sum_\alpha m_\alpha \bar{\bar{r}}_\alpha}_{0} \times (\bar{\omega} \times \bar{a}) = 0\,.$$

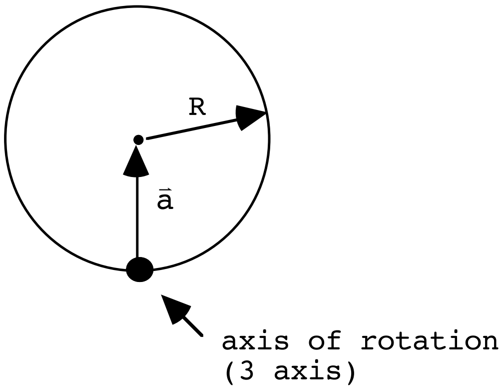
Therefore, we have

$$\sum_j J_{ij}\omega_j = M \sum_j \left[\delta_{ij} a^2 - a_i a_j\right]\omega_j + \sum_j I_{ij}\omega_j \,. \tag{11.24}$$

The coefficient of ωj on both sides of this equation must be the same:

$$\Rightarrow J_{ij} = I_{ij} + \overbrace{M\left[\delta_{ij}a^2 - a_i a_j\right]}^{\substack{\text{inertia tensor}\\ \text{for a point}\\ \text{mass at CM}}} \quad \text{“Parallel Axis Theorem”} \tag{11.25}$$

As an example, consider again a cylinder of radius R, but rotated from a point on it's edge rather than in the center as before.



We know from before that $I_{33} = \frac{1}{2}MR^2$. Since the vector $\vec{a}$, pointing from the 3 axis to the old ($\overline{3}$) axis has no components along this direction, we have $J_{33} = \frac{3}{2}MR^2$.

11.7 EULER ANGLES

We have written K.E. and angular momentum as

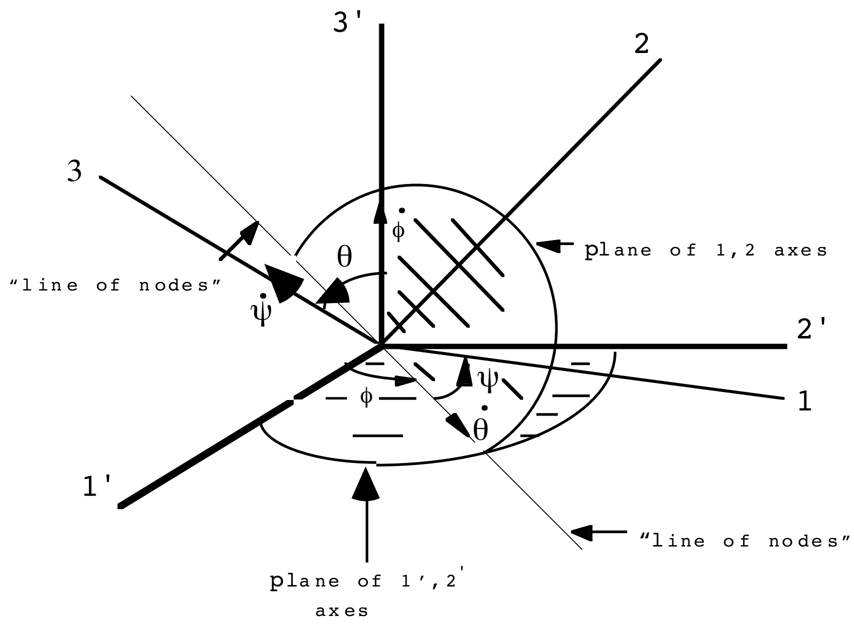
$$T = \frac{1}{2}\sum_{i,j}\omega_i I_{ij}\omega_j \;, \quad L_i = \sum_j I_{ij}\omega_j \;,$$

under the conditions that the body is rigid and that the origin of the body system is either at the center of mass or at a point for which $\dot{\vec{R}} = 0$. The analogous results for linear motion are of course

$$T = \frac{1}{2}\sum_i M\dot{r}_i^2 \;, \quad P_i = M\dot{r}_i \,.$$

There is a major difference in handling the two cases: When the $\dot{r}_i$ are integrated, we get linear coordinates that tell us, at each instant in time, the location of the particle or body. The ωi, however, when integrated are not quantities which give us the orientation of a rigid body in space. In order to describe the orientation, 3 angular quantities will be needed just as 3 linear quantities are needed to locate a body's origin. Many different choices or schemes are possible. A conventional and convenient choice are the “Euler angles”.

First, I will simply describe the situation. (Better pictures are available in other books.) Final situation:



θ, ψ, ϕ are the Euler angles. (As shown, all are positive.) They represent a choice of generalized coordinates useful in defining the Lagrangian for a moving rigid body.

General relation relating rotated coordinates:

$$x = \lambda x'. \qquad \text{(matrix notation)} \tag{11.26}$$

Represents a passive rotation. First, rotate positively (counterclockwise as viewed from above the 1',2' plane) through an angle θ about x_3':

$$x'' = \lambda_\phi x', \tag{11.27}$$

where

$$\lambda_\phi = \begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11.28}$$

Special notation:

$$c\phi \equiv \cos\phi, s\phi \equiv \sin\phi. \tag{11.29}$$

Now rotate (counterclockwise) by θ about x_1'':

$$\mathbf{x}''' = \lambda_\theta \mathbf{x}'' , \tag{11.30}$$

$$\lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{pmatrix} . \tag{11.31}$$

Next, rotate (counterclockwise) by ψ about x_3''':

$$\mathbf{x} = \lambda_\psi \mathbf{x}''' , \tag{11.32}$$

$$\lambda_\psi = \begin{pmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{11.33}$$

Put it all together:

$$\mathbf{x} = \lambda_\psi \left(\lambda_\theta \left(\lambda_\phi \mathbf{x}' \right) \right) = \left(\lambda_\psi \lambda_\theta \lambda_\phi \right) \mathbf{x}' . \tag{11.34}$$

Call $\lambda \equiv \lambda_\psi \lambda_\theta \lambda_\phi$. Explicitly,

$$\lambda = \begin{pmatrix} (c\psi c\phi - c\theta s\phi s\psi) & (c\psi s\phi + c\theta c\phi s\psi) & (s\psi s\theta) \\ (-s\psi c\phi - c\theta s\phi c\psi) & (-s\psi s\phi + c\theta c\phi c\psi) & (c\psi s\theta) \\ (s\theta s\phi) & (-s\theta c\phi) & c\theta \end{pmatrix}.$$

We may now think of the angular velocities $\dot{\vec{\psi}}, \dot{\vec{\theta}}, \dot{\vec{\phi}}$ as describing the instantaneous state of rotation of the body. [Directions come from infinitsmal rotations of ψ, θ, or ϕ and the right hand rule.] We would like to find the relationship between these quantities and the $\omega_{1,2,3}$. ($\vec{\omega}$ represents of course the instantaneous angular velocity projected on the body axes.)

First of all, it's obvious that

$$\dot{\vec{\psi}} = \dot{\psi}\,\hat{e}_3 . \tag{11.35}$$

What about $\dot{\vec{\theta}}$ and $\dot{\vec{\phi}}$? It's also clear from the picture that ($\dot{\vec{\theta}}$ is in the 1,2 plane)

$$\dot{\vec{\theta}} = \dot{\theta}\cos\psi\,\hat{e}_1 - \dot{\theta}\sin\psi\,\hat{e}_2 \tag{11.36}$$

Let's now find the components of $\dot{\vec{\theta}}$ using the Euler matrix. All we have to do is:

$$x = \lambda x' \quad \text{where} \quad x' = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix}.$$

It is easy to verify that

$$x = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta \\ \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta \end{pmatrix},$$

$$\Rightarrow \dot{\vec{\phi}} = \dot{\phi}\sin\psi\sin\theta\,\hat{e}_1 + \dot{\phi}\cos\psi\sin\theta\,\hat{e}_2 + \dot{\phi}\cos\theta\,\hat{e}_3 . \tag{11.37}$$

Adding it up gives

$$\vec{\omega} = \dot{\vec{\psi}} + \dot{\vec{\phi}} + \dot{\vec{\theta}},$$

$$\Rightarrow \vec{\omega} = \left(\dot{\phi} \sin \phi \sin \theta + \dot{\theta} \cos \psi, \dot{\phi} \cos \psi \sin \theta \right.$$

$$\left. -\dot{\theta} \sin \psi, \dot{\psi} + \dot{\phi} \cos \theta\right) \tag{11.38}$$

or

$$\begin{cases} \omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi, \\ \omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi, \\ \omega_3 = \dot{\psi} + \dot{\phi} \cos \theta. \end{cases}$$

11.9 EULER'S EQUATIONS OF MOTION

We are going to study the motions of rigid bodies using 2 methods. One, we already know, is the Lagrangian method. This is the reason Euler angles have been introduced. The Lagrangian approach is clear. The kinetic energy is

$$T = \frac{1}{2} \sum_{i,j} \omega_i I_{ij} \omega_j .$$

We now express $\omega_{1,2,3}$ in terms of the Euler angles θ, ψ, ϕ and their derivatives and use the Euler-Lagrange equations of motion,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) = 0,$$

where $q_i \in (\theta, \psi, \phi)$.

There is another approach possible which works directly with the $\omega_{1,2,3}$. Remember:

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rotating} + \vec{\omega} \times \vec{Q} .$$

Take our fixed and rotating axes origins at the CM of the particle and instantaneously coincident in direction. Let $\vec{Q} \rightarrow \vec{L}$,

$$\left(\frac{d\vec{L}}{dt}\right)_f = \left(\frac{d\vec{L}}{dt}\right)_r + \vec{\omega} \times \vec{L}.$$

But

$$\left(\frac{d\vec{L}}{dt}\right)_f = \vec{N} \quad \text{and} \quad \left(\vec{L}\right)_i = \sum_j I_{ij}\omega_j\,,$$

and also

$$\left(\frac{dL_i}{dt}\right)_r = \sum_j \frac{d}{dt}\left(I_{ij}\omega_j\right).$$

But since we recognize that the body axes are attached to the body

$$\frac{d}{dt} I_{ij}\omega_j = I_{ij}\dot{\omega}_j\,,$$

then

$$N_i = \sum_j I_{ij}\dot{\omega}_j + \sum_{j,k,\ell} \varepsilon_{ijk}\ \omega_j \left(I_{k\ell}\omega_\ell\right). \tag{11.39}$$

If we now pick our body axes along the principal axes, then $I_{ij}\ (i \neq j) = 0$, and

$$\sum_{j,k,\ell} \varepsilon_{ijk}\ \omega_j\omega_\ell\ \underbrace{I_{k\ell}}_{=I_k\delta_{k\ell}} = \sum_{j,k} \varepsilon_{ijk}\ \omega_j\omega_k\ I_k\ .$$

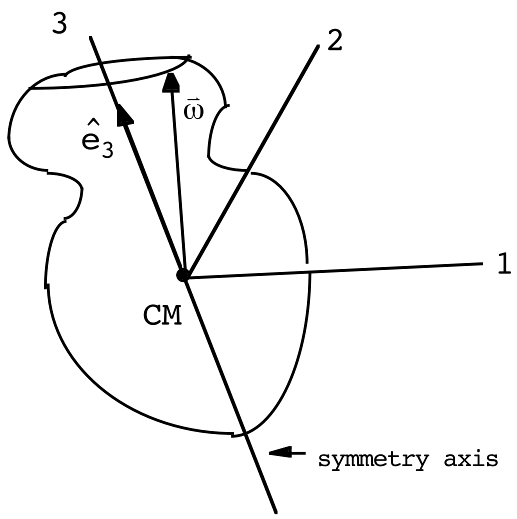
Written out explicitly, we have ("Euler's equations")

$$\left.\begin{aligned} N_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3\left(I_3 - I_2\right), \\ N_2 &= I_2\dot{\omega}_2 + \omega_3\omega_1\left(I_1 - I_3\right), \\ N_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2\left(I_2 - I_1\right). \end{aligned}\right\} \tag{11.40}$$

They are completely equivalent to the Lagrangian equations but in general have a different form when $\omega_{1,2,3}$ are expressed in terms of $\theta,\ \psi,\ \phi$. They are coupled first order differential equations. One strategy: solve for $\omega_{1,2,3}(t)$ and then invert the above relationships between $\omega_{1,2,3}$ and $\theta,\ \psi,\ \phi$ and their derivatives to solve for $\theta(t)$, $\psi(t)$, $\phi(t)$.

11.9 SYMMETRICAL TOP – EULER SOLUTION

Let's try carrying out this procedure for torque-free motion ($\vec{N} = 0$) for a body with $\underbrace{I_1 = I_2}_{I_{12}} \neq I_3$ (a "symmetrical top").



With no loss in generality, take $\omega_3 > 0$. We can realize such a situation in an orbiting space craft or a falling elevator, say.

Of course, if $\vec{\omega}$ is initially pointed along a principal axis (along 3 or anywhere in the 1,2 plane), then the motion is trivial. Then we simply have,

$$0 = I_3\dot{\omega}_3 \quad \text{or} \quad 0 = I_{12}\dot{\omega}_{1,2}$$

$$\Rightarrow \omega_3 = \textbf{constant} \quad \textbf{or} \quad \omega_{1,2} = \textbf{constant.}$$

In the general case, we have to solve:

$$\left.\begin{aligned} &I_{12}\dot{\omega}_1 + \omega_2\omega_3\left(I_3 - I_{12}\right) = 0, \\ &I_{12}\dot{\omega}_2 + \omega_3\omega_1\left(I_{12} - I_3\right) = 0, \\ &I_3\dot{\omega}_3 = 0. \end{aligned}\right\}$$

Notice immediately that ω_3 = const. in the general case also. We may now write the other two equations as

$$\dot{\omega}_1 + \Omega\omega_2 = 0,$$

$$\dot{\omega}_2 - \Omega\omega_1 = 0,$$

where (the "precession rate")

$$\underbrace{\Omega}_{\substack{\text{(either positive}\\ \text{or negative)}}} \equiv \underbrace{\left(\frac{I_3 - I_{12}}{I_{12}}\right)}_{\text{positive constant}} \omega_3 . \tag{11.41}$$

Then we have

$$\ddot{\omega}_1 + \dot{\omega}_2\Omega = \ddot{\omega}_1 + \omega_1\Omega^2 = 0. \tag{11.42}$$

Solution:

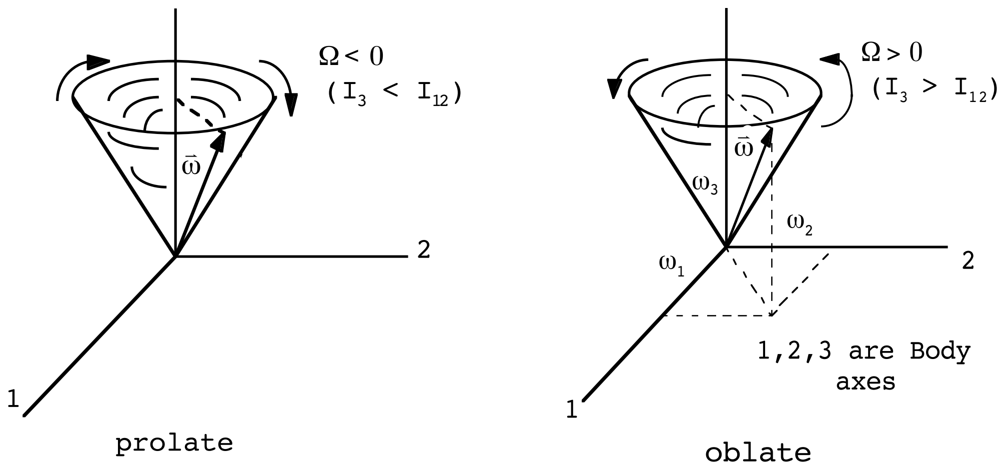
$$\omega_1(t) = A\cos\left(\Omega t + \alpha\right).$$

Since

$$\omega_2 = -\frac{1}{\Omega}\dot{\omega}_1,$$

$$\Rightarrow \omega_2 = A\sin(\Omega t + \alpha).$$

(A, α = constants; can take α =0 with no loss in generality.) We recognize these as the parametric representation of a circle since $A^2 = \omega_1^2 + \omega_2^2$. Describes (note that $|\vec{\omega}|$ =constant.):



$\vec{L}$, $\vec{\omega}$, $\hat{e}_3$ are all in the same plane:

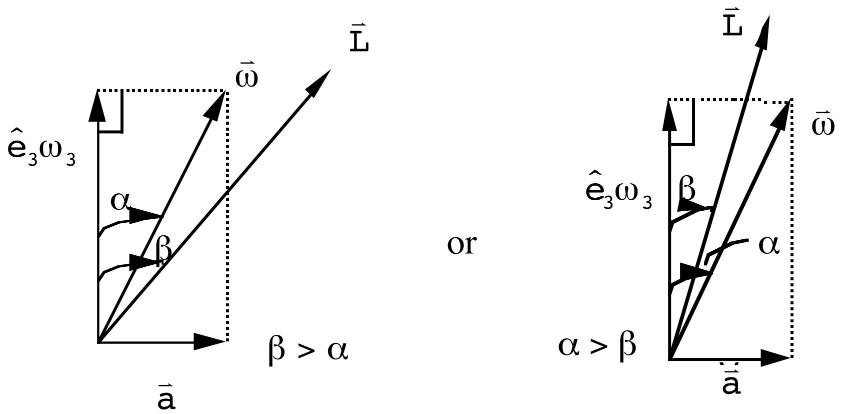
$$\vec{L} = I_{12}\left(\omega_1\hat{e}_1 + \omega_2\hat{e}_2\right) + I_3\hat{e}_3\omega_3 \;,$$

$$\dot{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3 \;,$$

so

$$\vec{L}\cdot\left(\vec{\omega}\times\hat{e}_3\right) = \vec{L}\cdot\left(-\omega_1\hat{e}_2 + \omega_2\hat{e}_1\right) = 0 \,.$$

Somewhat pitiful pictures:



Case 1: prolate Case 2: oblate

$\vec{L}$ is a constant in magnitude and direction in the above diagrams. We have

$$\tan\beta = \frac{\vec{L}\cdot\dfrac{\vec{a}}{a}}{L_3} = \frac{\vec{L}}{L_3}\cdot\left(\frac{\vec{\omega} - \omega_3\hat{e}_3}{\left|\vec{\omega} - \omega_3\hat{e}_3\right|}\right),$$

$$= \frac{I_{12}\omega_1^2 + I_{12}\omega_2^2}{I_3\omega_3}\cdot\frac{1}{\sqrt{\omega_1^2 + \omega_2^2}} \;,$$

$$\Rightarrow \tan\beta = \frac{I_{12}}{I_3}\frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_3} \,. \tag{11.43}$$

Clearly β is a constant in time. Also

$$\tan\alpha = \frac{|\vec{a}|}{\omega_3} = \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_3} \tag{11.44}$$

is also a constant. They are related by

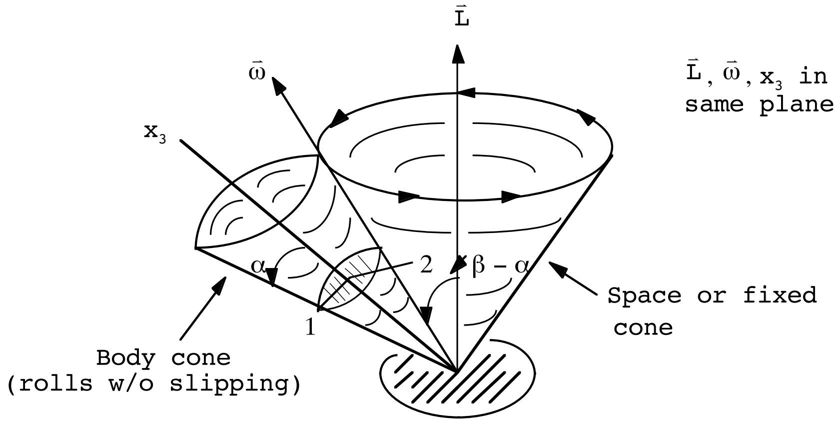
$$\frac{\tan\beta}{\tan\alpha} = \frac{I_{12}}{I_3}. \tag{11.45}$$

Therefore:

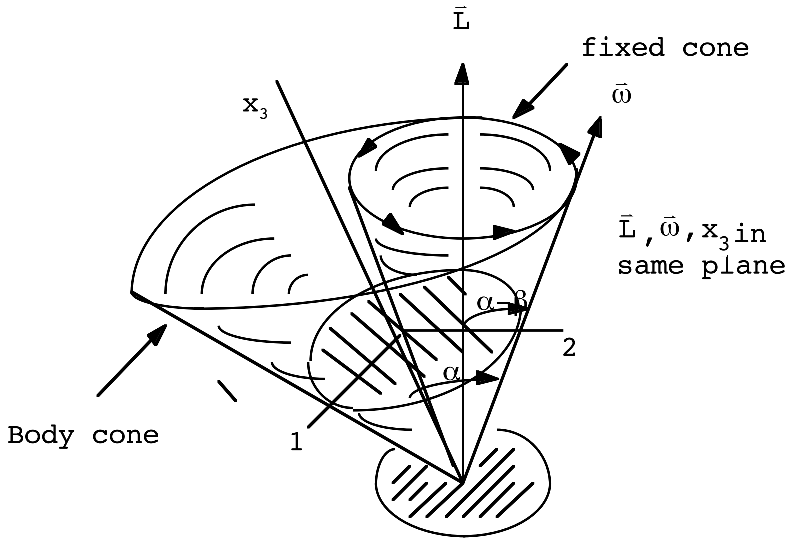
If $I_3 < I_{12}\ (\Omega < 0) \Rightarrow \beta > \alpha$, (Case 1: prolate)

If $I_3 > I_{12}\ (\Omega > 0) \Rightarrow \alpha > \beta$, (Case 2: oblate)

Motion is summed up in the following pictures:



Case 1: $\Omega < 0, \quad \beta > \alpha$



Case 2: $\Omega > 0, \quad \alpha > \beta$

Notice: $\vec{\omega}$ rotates clockwise w.r.t. 1,2 (body) axes in Case 1, whereas it rotates counterclockwise w.r.t. 1,2 axes in Case 2. Also notice if $I_{12} = I_3$, $\vec{L}$ and $\vec{\omega}$ point in the same direction and there is no precession. (The case for origin at CM of a sphere or cube.)

Now complete the discussion by getting $\theta(t)$, $\psi(t)$, $\phi(t)$ from $\omega_{1,2,3}(t)$. Remember:

$$\begin{cases} \omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \\ \omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi, \\ \omega_3 = \dot{\psi} + \dot{\phi} \cos\theta. \end{cases}$$

or, inverting,

$$\dot{\phi} = \frac{\omega_1 \sin\psi + \omega_2 \cos\psi}{\sin\theta}, \tag{11.46}$$

$$\dot{\theta} = \omega_1 \cos\psi - \omega_2 \sin\psi, \tag{11.47}$$

$$\dot{\psi} = \omega_3 - \cot\theta \left(\omega_1 \sin\psi + \omega_2 \cos\psi\right). \tag{11.48}$$

If we take x_3' along $\bar{L}$, then we identify

$$\theta = \beta. (b = \text{const.}) \tag{11.49}$$

Therefore $\dot{\theta} = 0$ and (from (11.47))

$$0 = \omega_1 \cos\psi - \omega_2 \sin\psi \,,$$

$$\Rightarrow 0 = A\cos(\Omega t)\cos\psi - A\sin(\Omega t)\sin\psi \,,$$

or

$$0 = A\cos(\Omega t + \psi).$$

If A ≠ 0, then we may choose ($\psi(t)$ gives rotation angle w.r.t. line of nodes)

$$\psi(t) = -\Omega t + \frac{\pi}{2}, \tag{11.50}$$

without loss in generality. And, from (11.46),

$$\dot{\phi} = \frac{A}{\sin\theta}\left(\cos(\Omega t)\sin\psi + \sin(\Omega t)\cos\psi\right),$$

$$\Rightarrow \dot{\phi} = \frac{A}{\sin\theta}\sin\underbrace{\left(\Omega t + \psi(t)\right)}_{\pi/2} = \frac{A}{\sin\theta},$$

$$\Rightarrow \phi(t) = \frac{A}{\sin\beta}t + \overset{\text{(choice)}}{\overset{0}{\cancel{C}}} \quad (\theta = \beta) \tag{11.51}$$

Eq.(11.48) is now satisfied also since

$$-\Omega \overset{?}{=} \omega_3 - \cot\theta\left(\frac{A}{\sin\theta}\sin\theta\right),$$

$$\cot\theta = \cot\beta = \frac{I_3\omega_3}{I_{12}\sqrt{\omega_1^2 + \omega_2^2}},$$

$$(11.41) \Rightarrow -\left(\frac{I_3}{I_{12}} - 1\right)\omega_3 \overset{?}{=} \omega_3 - A\frac{I_3\omega_3}{I_{12}\sqrt{\omega_1^2 + \omega_2^2}},$$

$$\Rightarrow A = \sqrt{\omega_1^2 + \omega_2^2}. \ \checkmark$$

Thus, the time dependence of the Euler angles is given by

$$\theta = \beta, \ \psi(t) = -\Omega t + \frac{\pi}{2}, \ \phi(t) = \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\sin\beta} t .$$

$\phi(t)$ may also be written as:

$$\phi(t) = \frac{\sin\alpha}{\sin(\alpha - \beta)} \Omega t ,$$

where α, β are defined above. Thus, $\frac{\sin\alpha}{\sin(\alpha - \beta)} \Omega$ is the precession rate of the line of nodes.

11.10 SYMMETRICAL TOP – LAGRANGIAN SOLUTION

We have used Euler's equations to first get $\omega_{1,2,3}(t)$, then use these to get $\theta(t)$, $\psi(t)$, $\phi(t)$. Other way of proceeding: Lagragian using θ, ψ, ϕ from the start. Now do the same problem (force free symmetrical top) from the Lagrangian point of view. We have

$$T = \frac{1}{2} \sum_{i,j} \omega_i I_{ij} \omega_j .$$

Again, pick principal axes with symmetry axis of body along x_3:

$$T = \frac{1}{2} I_{12} \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2 .$$

But now use earlier expressions for $\omega_{1,2,3}$ in terms of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$:

$$T = \frac{1}{2}\left[I_{12}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2\right]. \tag{11.52}$$

The Lagrangian equations are (L = T here):

$$\frac{\partial T}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = 0 , \tag{11.53}$$

$$\frac{\partial T}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = 0 , \tag{11.54}$$

$$\frac{\partial T}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = 0 , \tag{11.55}$$

We have

$$\begin{cases} \dfrac{\partial T}{\partial \dot\theta} = I_{12}\dot\theta\,, \\[2mm] \dfrac{\partial T}{\partial \dot\phi} = I_{12}\dot\phi\sin^2\theta + I_3\left(\dot\psi + \dot\phi\cos\theta\right)\cos\theta\,, \\[2mm] \dfrac{\partial T}{\partial \dot\psi} = I_3\left(\dot\psi + \dot\phi\cos\theta\right), \end{cases}$$

$$\begin{cases} \dfrac{\partial T}{\partial \theta} = I_{12}\dot\phi^2\sin\theta\cos\theta - I_3\dot\phi\left(\dot\psi + \dot\phi\cos\theta\right)\sin\theta \\[2mm] \dfrac{\partial T}{\partial \phi} = \dfrac{\partial T}{\partial \psi} = 0\,. \end{cases}$$

Putting the pieces together gives us the Lagrangian equations of motion:

$$(11.53) \Rightarrow I_{12}\ddot\theta - I_{12}\dot\phi^2\sin\theta\cos\theta + I_3\dot\phi\left(\dot\psi + \dot\phi\cos\theta\right)\sin\theta = 0, \qquad (11.56)$$

$$(11.54) \;\Rightarrow\; \frac{d}{dt}\left[I_{12}\dot{\phi}\sin^2\theta + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta\right] = 0, \tag{11.57}$$

$$(11.55) \;\Rightarrow\; \frac{d}{dt}\left[I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\right] = 0. \tag{11.58}$$

Euler's equations can be recovered from these. For example, substitute in (11.58):

$$\dot{\psi} = \omega_3 - \cot\theta\left(\omega_1 \sin\psi + \omega_2 \cos\psi\right),$$

$$\dot{\phi} = \frac{1}{\sin\theta}\left(\omega_1 \sin\psi + \omega_2 \cos\psi\right),$$

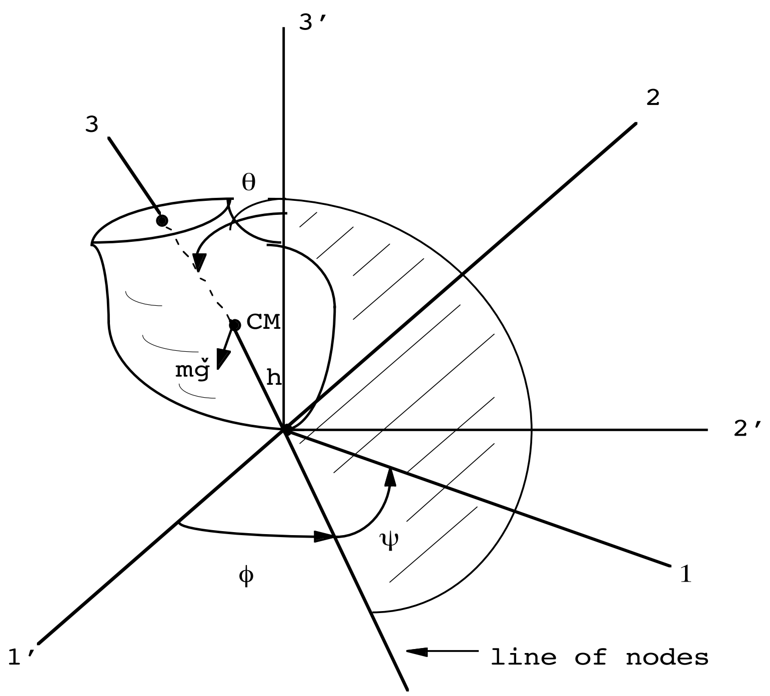
$$\Rightarrow \dot{\psi} + \dot{\phi}\cos\theta = \omega_3 - \cot\theta\,(\omega_1 \sin\psi + \omega_2 \cos\psi \\ -\omega_1 \sin\psi - \omega_2 \cos\psi).$$

Thus, Eq.(11.58) just says

$$I_3\dot{\omega}_3 = 0,$$

which is one of the Euler equations above. Likewise, (11.56) and (11.57) above are equivalent to the other two Euler equations (after some manipulations).

Can also apply the Lagrangian approach in describing the symmetrical top with one point fixed:



Same T as above, except now $L = T - U$, where $U = mgh \cos\theta$. From the previous results, one can show that the total energy, $E = T + U$, can be written as

$$E = \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} I_{12} \dot{\theta}^2 + U_{eff}(\theta), \tag{11.59}$$

where the effective potential, $U_{eff}(\theta)$, is given by,

$$U_{eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_{12} \sin^2\theta} + mgh \cos\theta, \tag{11.60}$$

where (11.57), (11.58) give rise to the conserved conjugate momentums,

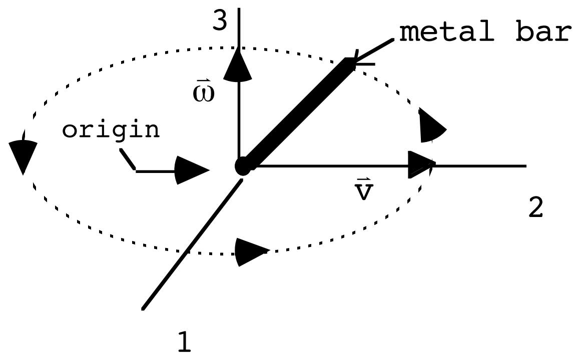
$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = I_{12} \dot{\phi} \sin^2\theta + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta, \tag{11.61}$$

$$p_\psi \equiv \frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right). \tag{11.62}$$

The minimum of this effective potential gives the angle of inclination at which the top precesses steadily.

11.11 PROBLEMS

1.



A thin solid metal bar (ρ = const.) of mass m and length l is rotated uniformly about one end at an angular velocity, w, directed along the 3 axis.

a) Compute the kinetic energy.
b) The origin in (a) is given a velocity, $\vec{v}$, along the 2 axis. Now compute the kinetic energy. (Be careful!)

2. (a) Given that the potential energy of a small mass m_a is given by $V_a = gm_a x_a$, where x_a is the height of m_a, show that the total potential energy of an arbitrary object in the gravitational field is given by V = gMX, where X is x-coordinate center of mass coordinate, $X \equiv (\vec{R}_{CM})_x$, and M is the total mass.

(b) Show that the total torque on the object, $\vec{N}$, is given by

$$\vec{N} = \vec{R}_{CM} \times \vec{F},$$

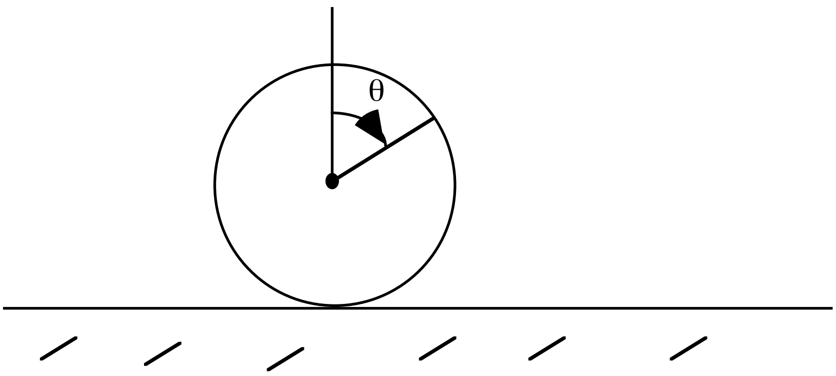
where $\vec{R}_{CM}$ locates the center of mass and $\vec{F}$ is the total force.

3. (a) A thin uniform rod of length b and mass m stands vertically upright on a rough floor and then tips over. What is the rod's angular velocity just before it hits the floor?

(b) Repeat for a rod initially standing upright on a smooth floor (no friction). Find the final angular velocity about the CM just before striking the floor.

(c) You should have gotten the same angular velocity for both (a) and (b) parts. Why?

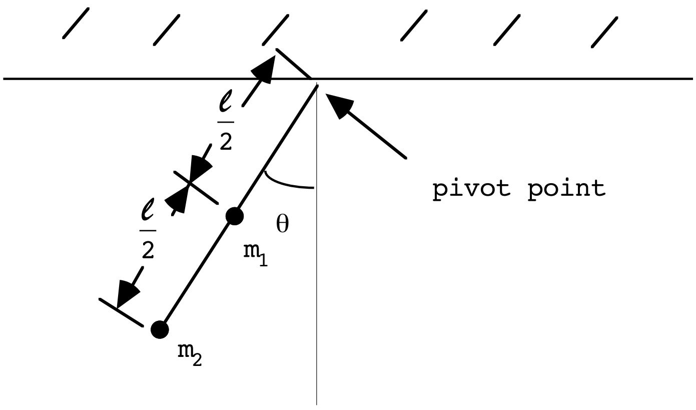
4. A solid disk (ρ = const., radius = R, mass = M) rolls without slipping on the ground at a constant rate, $\omega = \dot{\theta}$ = const. Compute it's kinetic energy two ways.



a) By using Eq.(11.5) of the text.
b) By using the fact that the point of contact between the disk and ground acts as the *instantaneous* axis of rotation. (You will have to use the parallel axis theorem to do this part.)

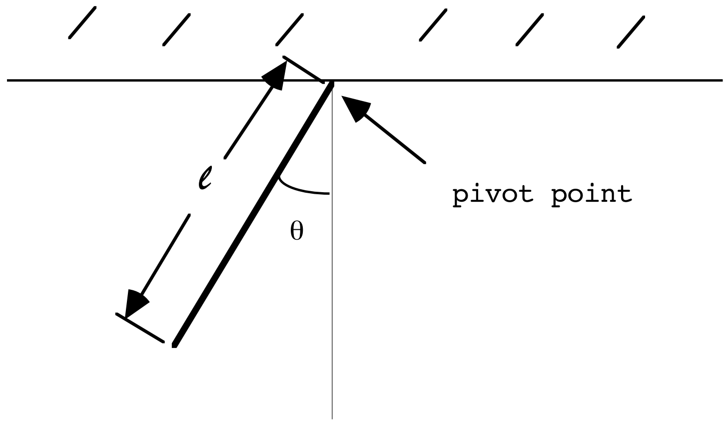
Do you get the same answers for (a) & (b)?

5. A pendulum consists of two point masses, m_1 and m_2, attached to a completely rigid, massless bar of length ℓ. m_1 is attached halfway down the bar, m_2 is at the end.



a) Find the bar's moment of inertia about the pivot point.
b) Assuming small oscillations, find the bar's period of motion.

6. A pendulum consists of a thin bar of mass m and length ℓ attached to a pivot point as shown. It swings in a plane.



a) Find the bar's moment of inertia about the pivot point.
b) Assuming small oscillations, find the bar's period of motion. Does it swing faster or slower than a pendulum of the same length with mass m concentrated at it's end?

7. (a) A "physical pendulum" consists of an arbitrary object of mass M whose moment of inertia about the pivot point is I. (Motion still confined to a plane.) Given that R is the distance from the center of mass of the object to the pivot point, show that this pendulum has the same small oscillation period as a simple pendulum of length $L = \frac{I}{MR}$.

(b) Find the length of simple pendulums with the same period as problems 11.5 and 11.6.

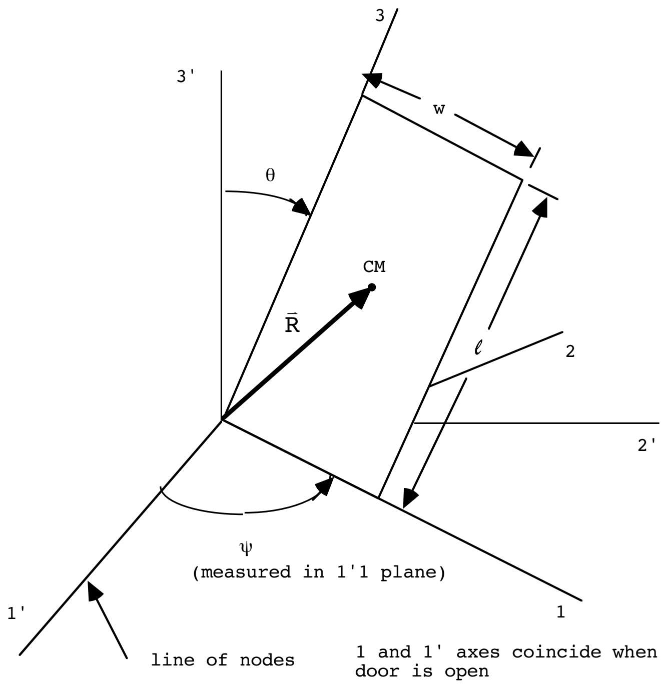
8. The moment of inertia tensor, I, for some object has the form,

$$I = \begin{pmatrix} X & M & 0 \\ M & X & 0 \\ 0 & 0 & Y \end{pmatrix},$$

where X, Y and M are some given real numbers (X, Y > 0 and |M| < X), find:

a) the three moments of inertia, I'_i, i=1,2,3.
b) the angular rotation eigenvectors, $\bar{\omega}^i$, associated with the principle axes.

9. ("Slamming door problem") A door is constructed of a thin homogeneous slab of material; it has a width w and a height ℓ. You are given that the line of hinges bends inward 2° with respect to the vertical, in the 3'–2' plane; see the figure. Assuming frictionless hinges, what is the angular velocity of the door just before it closes if it starts from rest after being opened 90°? [Hint: Find the Euler angles which transform from the body to the fixed axes and conserve energy.]



10. (a) Given principal axes for which the angular momentum vector may be written

$$\vec{L} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3 ,$$

where $\hat{e}_{1,2,3}$ are body frame unit vectors, show that the components of $\vec{L}$ along the $\hat{\dot{\theta}}$, $\hat{\dot{\phi}}$ and $\hat{\dot{\psi}}$ directions are

$$(I_1 = I_2 \equiv I_{12} \neq I_3 \text{ case})$$

$$L_\theta = I_{12}\ \dot{\theta} ,$$

$$= I_{12}\ \dot{\phi}\ \sin^2\theta + I_3\ (\dot{\psi} + \dot{\phi}\cos)\cos\theta ,$$

$$L_\psi = I_3\ (\dot{\psi} + \dot{\phi}\cos)\ \theta .$$

(b) Alternatively, given components of $\vec{\omega}$ along the $\hat{\dot{\theta}}$, $\hat{\dot{\phi}}$ and $\hat{\dot{\psi}}$ directions as ω_θ, ω_ϕ, and ω_ψ respectively, show that one may write

$$L_\theta = I_{12}\ \omega_\theta ,$$

$$L_\phi = I_{12}\ (\omega_\phi - \omega_\psi\cos\theta) + I_3\ \omega_\psi\cos\theta ,$$

$$L_\psi = I_3\ \omega_\psi .$$

11. Eqs.(11.56) and (11.57) are supposedly equivalent to the Euler equations:

$$I_{12}\,\dot{\omega}_1 + (I_3 - I_{12})\,\omega_2\omega_3 = 0, \qquad (1)$$

$$I_{12}\,\dot{\omega}_2 + (I_{12} - I_3)\,\omega_3\omega_1 = 0. \qquad (2)$$

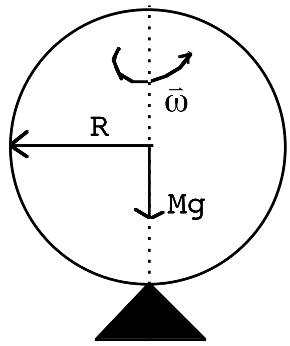
By using the expression for $\omega_{1,2,3}$ in terms of Euler angles, show that Eqs.(11.56) and (11.57) may be recovered from the above. [Hint: Consider (1) cos ψ – (2) sin ψ and (1) sin ψ + (2) cos ψ.]

12. Show the expression for the effective potential for the symmetrical top, Eq.(11.60),

$$U_{eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_{12}\sin^2\theta} + mgh\cos\theta,$$

follows from the expression for the total energy (T in Eq.(11.52) and U = mgh cosθ) and the definitions of the cyclic momentums, p_ϕ and p_ψ, given in (11.61), (11.62).

13. A solid sphere (ρ = const.) is made into a fixed point top by supporting it on a point as shown. Gravity is acting downward.



Find the minimum angular frequency, ω_{min}, such that the sphere is stable in a vertical position. (Use the result of #12 above.)

14. Consider a thin rod for which I_3 = 0, making the Euler angle ψ irrelevant. Show that the effective potential for this system has a minimum at an angle θ_0 given by

$$\cos\theta_0 = -\frac{mgh}{I_1\dot{\phi}^2}\ .$$

Other Problems

15. Given that the 1',2',3' axes are principal axes with moments of inertia I'_{11}, I'_{22}, I'_{33} respectively, find the moment of inertia along the new 3 direction, I_{33}, in the Euler angle diagram, p.11.20 of the text (angles are θ, ψ and ϕ), using the transformation law of the I_{ij} under rotations.

16. Show that the ratio of the magnitude of the instantaneous angular velocity, $|\bar{\omega}|$, to the precession rate, Ω, for a symmetrical top undergoing torque-free motion, is given by

$$\frac{|\bar{\omega}|}{\Omega} = \frac{\sin\beta}{\sin(\alpha - \beta)},$$

where α is the angle of the vector $\bar{\omega}$ with respect to the 3 (body) axis and β is the angle of the instantaneous angular momentum, $\bar{L}$, with respect to the same axis.

17. A "spherical pendulum" is a simple pendulum that is unconstrained in it's angular motion; it can move freely about it's pivot point. Consider a spherical pendulum of length L acted on by gravity with a mass M attached to it's end. The mass of the attachment of length L is negligible.

a) Derive the Euler-Lagrange equations for the pendulum in the spherical coordinates θ (polar angle) and ϕ (azimuthal angle). Take θ=0 to represent the equilibrium position. Show that the z-component of the angular momentum, ℓ_z, is conserved.

b) For uniform circular motion in f when θ =const. ≠ 0, show that the period of the motion may be written as

$$T = 2\pi \left(\frac{L \cos \theta}{g} \right)^{1/2} .$$

c) Consider general motions of the pendulum subject only to the condition that θ << π/2. Defining the total energy of the system, E, relative to θ=0, show that the turning points of the θ motion are

$$\theta_t \approx \sqrt{\left(\frac{E}{MgL} \right) \pm \sqrt{\left(\frac{E}{MgL} \right)^2 - \frac{\ell_z^2}{gM^2L^3}}} .$$

18. A symmetric top ($I_1 = I_2 \equiv I_{12}$) of mass m and constants of the motion p_ψ and p_ϕ precesses steadily, $\dot{\phi}$ = const. Let $p_\psi = p_\phi \equiv p$. h is the distance from the CM to the fixed tip.

a) Find the angle, θ_0, at which the precession occurs.

b) Show the precession rate is

$$\dot{\phi} = \sqrt{\frac{mgh}{I_{12}}} .$$

19. (a) A thin uniform rod of length b and mass m stands *almost* vertically upright on a rough floor and then tips over without it's end slipping on the floor. The initial tipping angle is θ_0<<1 with respect to the vertical. Find the amount of time, T, that the rod takes to fall over and hit the floor.

(b) The same problem as in (a) but with a smooth, frictionless floor.

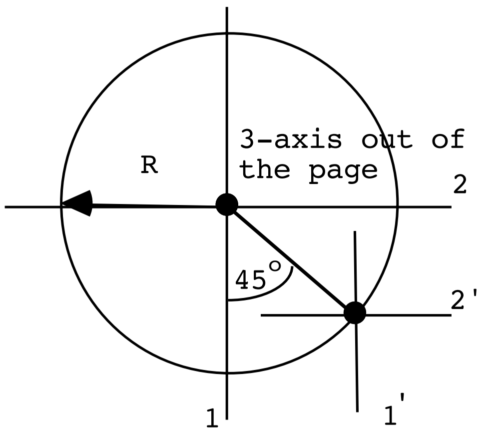
20. (a) Given a set of principal axes and all axes parallel to them for some object, show that the sum of the moments of inertia is minimized when the origin is at the center of mass.

(b) An object has principal axes for which $I_{11} > I_{22} > I_{33}$. Characterize all the points which are also principal axes for axes parallel to the original set.

Extra: Using the concept of eigenvalue degeneracy (for moments of inertia) and the parallel axis theorem, find a larger set of axes which are also principal axes for the object in (b), but which are not parallel to the original set.

21. Consider the symmetric top $(I_{11} = I_{22} \neq I_{33})$ with one point fixed. We briefly discussed how this system can be solved with a Lagrangian approach. However, now attempt to write Euler's equations for this system. Can it be done? Remember the components of the equations refer to the body system. If these equations can be formulated, discuss the manner in which they may be solved (but do not attempt it!). If not, tell me why not.

22. Consider a solid cylinder of mass M, length t, and radius R of uniform density, ρ. Let the 1, 2 plane be located at the top of the cylinder. (This is a little different from the situation in the text on p.11.5.)



a) Find the elements of the inertia tensor with respect to the unprimed axes.
b) Find the elements of the inertia tensor with respect to the primed axes.

12 COUPLED OSCILLATIONS

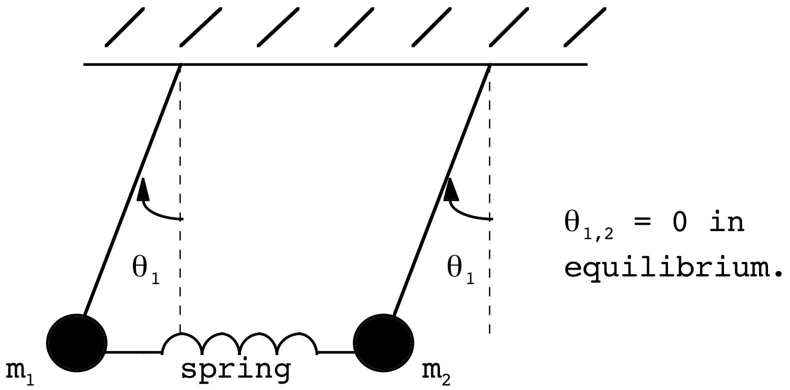
12.1 COUPLED DYNAMICAL EQUATIONS

In a number of cases we have derived *coupled* linear differential equations. We have treated such systems as special cases, but it is now time to develop some general techniques for handling them. Mathematically, we will encounter here both the use of complex numbers (which were used in the Focault pendulum discussion and various differential equation solutions) as well as the eigenvalue-eigenvector matrix algebra of the last chapter. Examples of systems these techniques will cover: conservative, linear mechanical or electrical oscillations, molecular vibrations, approximate planetary motions, and many more. We will also use stability analysis and the concept of generalized coordinates. I will first introduce the theory and then will work out a number of examples to illustrate the general techniques.

Let us consider a system with generalized coordinates q_j (j = 1,…, n; n = no. of "degrees of freedom"), related formally to the $x_{\alpha i}$ by

$$x_{\alpha i} = x_{\alpha i}(q_j) \Rightarrow \frac{\partial x_{\alpha i}}{\partial t} = 0. \tag{12.1}$$

Usually, the q_j we will consider will be measured from *equilibrium* positions or lengths. For example,



Let us also only consider $U = U(q_j)$. This means we are limiting our discussion to conservative systems for which $H = T + U = \text{const.}$ in time. Another limitation of the following discussion will be the assumption that the system is *linear*. That is, we will only consider potentials that are *quadratic* in the generalized coordinates, or are approximately quadratic. For small oscillations, we have

$$U(q_1, q_2, ..., q_n) = U_0 + \sum_k \frac{\partial U}{\partial q_k}\Big|_0 q_k + \frac{1}{2}\sum_{j,k} \underbrace{\frac{\partial^2 U}{\partial q_j \partial q_k}}_{\substack{=A_{jk}=A_{kj} \\ \text{(constants)}}}\Big|_0 q_j q_k + \ldots, \tag{12.2}$$

where higher order terms will be neglected. In addition, we will assume or require that (q_{10}, q_{20},… etc. are equilibrium q_i's):

1. $U_0 = U(q_1 = q_{10},\ q_2 = q_{20}, ...) = 0.$
2. $\frac{\partial U}{\partial q_k}\Big|_0 = 0$ for each k.
3. $\frac{\partial^2 U}{\partial Q^2}\Big|_0 > 0$, where Q is any linear combination of the generalized coordinates, q_i.

Condition 1 is no restriction in the applicability of the analysis since the absolute value of the potential U is always arbitrary up to an overall constant. Conditions 2 & 3 above are just conditions for stable equilibrium in a system with n degrees of freedom. Notice that condition 3 above (which is a sufficient, rather than necessary, condition for stability) is equivalent to the requirement that $U(q_i) > 0$ for quadratic potentials, given $U_0 = 0$. We can show this as follows. Assume (i = 1,…, n)

$$Q = \sum_i x_i q_i \,, \tag{12.3}$$

where the x_i are arbitrary constants. Then

$$\frac{\partial^2 U}{\partial Q^2} = \frac{\partial}{\partial Q} \sum_i \frac{\partial U(q_i)}{\partial q_i} \frac{\partial q_i}{\partial Q} \,,$$

but $\dfrac{\partial q_i}{\partial Q} = \dfrac{1}{x_i}$, so

$$\frac{\partial^2 U}{\partial Q^2} = \frac{\partial}{\partial Q} \sum_i \frac{\partial U(q_i)}{\partial q_i} \frac{1}{x_i} = \sum_{i,j} \frac{\partial^2 U}{\partial q_i \partial q_j} \frac{1}{x_i x_j} \,. \tag{12.4}$$

Condition 3 now gives,

$$\sum_{i,j} \frac{\partial^2 U}{\partial q_i \partial q_j}\Big|_0 \frac{1}{x_i x_j} > 0 \,. \tag{12.5}$$

(The sum in (12.5) is only over the nonzero x_i values.) Since the x_i are arbitrary (but nonzero) constants, this is the same as

$$U(\tilde{q}_i) > 0 \,, \tag{12.6}$$

where $\tilde{q}_i = \dfrac{1}{x_i}$ (see Eq.(12.2)).

Because of our condition $\dfrac{\partial x_{\alpha i}}{\partial t} = 0$, we know from Chapter 7 for a system of point particles, for example, that we may write (the m_{jk} here were called a_{jk} there)

$$T = \frac{1}{2} \sum_{j,k} m_{jk}\, \dot{q}_j \dot{q}_k \,, \tag{12.7}$$

where

$$m_{jk} = \sum_{\alpha,i} m_\alpha \frac{\partial x_{\alpha i}}{\partial q_j} \frac{\partial x_{\alpha i}}{\partial q_k} = m_{kj} \,. \tag{12.8}$$

Note that in general $m_{jk} = m_{jk}(q_i)$ (functions of the q_i), unlike the A_{jk} above, which are just constants. However, consistent with the restriction to quadratic terms in U, we will restrict ourselves to quadratic terms in the $\dot{q}_j$. This means that we will expand

$$m_{jk}(q_i) = m_{jk}\big|_0 + \underbrace{\sum_\ell \frac{\partial m_{jk}}{\partial q_\ell}\bigg|_0 q_\ell + \ldots,}_{\text{neglect}} \tag{12.9}$$

and neglect everything except the first term, which is assumed non-zero. The reason is that terms like $\sim q_\ell \dot{q}_j \dot{q}_k$ in the Euler-Lagrange equations produce nonlinear terms in the differential equations. (Neglecting such nonzero terms give additional restrictions on what is meant by "small oscillations".) Notice also that, in general

$$\frac{\partial T}{\partial q_\ell} = \frac{1}{2}\sum_{j,k} \frac{\partial m_{jk}}{\partial q_\ell}\dot{q}_j \dot{q}_k,$$

so that with (12.9) we have

$$\frac{\partial T}{\partial q_\ell} = 0.$$

After these restrictions, the general form of our Lagrangian in generalized coordinates, q_i, is

$$L = T - U = \frac{1}{2}\sum_{j,k} \underset{\substack{\uparrow \\ \text{real constants}}}{m_{jk}}\, \dot{q}_j \dot{q}_k - \frac{1}{2}\sum_{j,k} \underset{\uparrow}{A_{jk}}\, q_j q_k \,. \tag{12.10}$$

Equations of motion are given by

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0. \tag{12.11}$$

We have

$$\frac{\partial U}{\partial q_k} = \sum_j A_{jk}\, q_j \,, \qquad (A_{jk} = A_{kj}) \tag{12.12}$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_j m_{jk}\, \dot{q}_j \,. \qquad (m_{jk} = m_{kj}) \tag{12.13}$$

$$\Rightarrow \sum_j \left(A_{jk}\, q_j + m_{jk}\, \ddot{q}_j\right) = 0\,. \tag{12.14}$$

Although we have derived these equations in the context of particle oscillations, the small oscillations of many realistic rigid body systems can be so characterized as long as the Lagrangian may be put into the form of (12.10) above.

12.2 EIGENVALUE/EIGENVECTOR SOLUTION

We will use complex number analysis to simplify the solution of this sytem of equations. Under conditions 1, 2, 3 above, we know that the solutions are oscillations. Therefore assume

$$q_j(t) = a_j\, e^{i\omega t} \tag{12.15}$$

complex (involves two undetermined constants) — real frequency

where the *real* part of the right hand side is understood. Plugging this back above, we find

$$\sum_j \left(A_{jk} - \omega^2 m_{jk}\right) a_j = 0\,. \tag{12.16}$$

[This is almost the same form as the equations which determine principal axes:

$$\sum_j \left(I_{ij} - I'\delta_{ij}\right)\omega_j = 0\,.]$$

Again, the condition that there is a nontrivial solution for the a_j is that the determinant of the matrix $(A_{ij} - \omega^2 m_{ij})$ be zero:

$$\det \begin{pmatrix} A_{11} - \omega^2 m_{11} & A_{12} - \omega^2 m_{12} \ldots \\ A_{21} - \omega^2 m_{21} & A_{22} - \omega^2 m_{22} \ldots \\ \vdots & \vdots \end{pmatrix} = 0. \tag{12.17}$$

this gives an $n^{\underline{th}}$ order equation in ω^2, the roots of which will be labeled at ω_r^2, $r = 1, 2, \ldots n$. These are called the *eigenfrequencies* or *characteristic frequencies*. In general, the above is more difficult than solving the analogous equation for the principal moments of inertial since the ω^2 terms (the analog of the priciple moments, I') enter off-diagonal elements of the matrix also. However, one is solving the same *mathematical* problem in either case.

Analogy:

	Inertia tensor	Coupled Equations
eigenvalues:	I_i	ω_r^2
	(i = 1,2,3)	(r = 1,2,...,n)
	(principle axes)	(oscillatory modes)
eigenvectors:	$\vec{\omega}^i$ (ω_j^i)	$\vec{a}^r$ (a_j^r)
	j = {1,2,3}	j = {1,2,...,n}
	(vector components)	(vector components)

Just as there are directions in physical space which render the inertia tensor diagonal, there are directions in mode space, associated with the eigen frequencies ω_r^2 for which the motions uncouple. We get the $\vec{a}^r$ by the same procedure as in the last chapter: substitute a known in the algebraic equations, and then solve for the ratios

$$a_1^r : a_2^r : a_3^r : \ldots : a_n^r$$

for a given eigenvalue, ω_r^2. As before, the overall normalization of the $\vec{a}^r$ (chosen real) are arbitrary. This makes physical sense here also since this corresponds to the amplitude of the motion, which is not determined by the equations of motion but by the initial condition. Since there are n $\vec{a}^r$'s, we can require n condtions to arbitrarily normalize them. In the inertia tensor case, we required that

$$\vec{\omega}^i \cdot \vec{\omega}^i = 1\,,\ i = 1, 2, 3.$$

Here, we require,

$$\sum_{j,k} m_{jk}\, a_j^r a_k^r = 1\,,\ r = 1, 2, \ldots n. \tag{12.18}$$

(That this combination is always positive can be shown from the positivity of T, the kinetic energy.) Also, we can prove that (just a generalization to an n-dimensional space of the similar proof in Ch.11 for the inertia tensor)

$$\sum_{j,k} m_{jk}\, a_j^r a_k^s = 0\,,\quad (r \neq s) \tag{12.19}$$

for $\omega_r^2 \neq \omega_s^2$, and that for degenerate roots ($\omega_r^2 = \omega_s^2$) we may always choose this equation to hold. The above two conditions can be written together as

$$\sum_{j,k} m_{jk}\, a_j^r a_k^s = \delta_{rs} \,. \tag{12.20}$$

Also, we can show that the ω_r^2 are all real. In fact one can show, under conditions 1,2,3 above, that $\omega_r^2 > 0$.[1] Without loss of generality, one may further choose $\omega_r > 0$.

The general solution is now given by (real part still understood)

$$q_j(t) = \sum_{r=1}^{n} \beta_r\, a_j^r\, e^{i\omega_r t} \,, \tag{12.21}$$

where the a_j^r are now all determined (up to an overall sign) and β_r are so called "scale factors" which, in general, are complex. (We still have not built in the initial conditions, so we still need two arbitwrary constants for a second order differential equation.) The β_r are the amplitudes associated with the r[th] eigenmode as determined by the initial conditions. If only a single β_r is nonzero[1],

$$\beta_r = 0,\ r \neq k,\ \beta_k \neq 0,$$

then only a single eigenmode of the system has been excited and the solution of the motion is particularly simple.

$$q_j(t) = \beta_k\, a_j^k\, e^{i\omega_k t} \quad \text{(no k sum)}$$

Or, introducing the "normal coordinates",

$$n_k \equiv \beta_k\, e^{i\omega_k t} \,, \tag{12.22}$$

we have

$$q_j(t) = n_k\, a_j^k \,. \tag{12.23}$$

In general,

$$q_j(t) = \sum_k n_k\, a_j^k \,. \tag{12.24}$$

Expressed in normal coordinates, one has (using the orthogonality condition, (12.20), as well as (12.16))

$$T = \frac{1}{2}\sum_r \dot{n}_r^2 \;,\; U = \frac{1}{2}\sum_r \omega_r^2 n_r^2 \,,$$

$$\frac{\partial L}{\partial n_r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{n}_r}\right) = 0\,,$$

$$\Rightarrow \ddot{n}_r + \omega_r^2 n_r = 0. \qquad (12.25)$$

This in turn shows that the energy associated with each normal mode is constant since for the $r^{\underline{th}}$ mode,

$$\int dt\,\dot{n}_r \times \left(\ddot{n}_r + \omega_r^2 n_r = 0\right),$$

$$\Rightarrow E = \text{constant} = \frac{1}{2}\dot{n}_r^2 + \frac{1}{2}\omega_r^2 n_r^2$$

$$= T_r + U_r \;. \qquad (12.26)$$

Final cookbook recipe for solving a system for small oscillations:

1. Write the Lagrangian, L, of the system in terms of generalized coordinates, qi, and find the Ajk and mjk either by using the explicit formulas $\left(A_{jk} = -\left.\frac{\partial^2 L}{\partial q_j \partial q_k}\right|_0 ,\; m_{jk} = \left.\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k}\right|_0\right)$ or by comparing to the form of the Lagrangian, Eq.(12.10).

2. Form the matrix $\left(A_{jk} - \omega^2 m_{jk}\right)$ and find the n eigenvalues, ω_r^2.

3. Determine the ratios $a_1^r : a_2^r : a_3^r : \ldots : a_n^r$ and normalize the a_i^r according to Eq.(12.20).

4. Write the general solution for the q_i as in Eq.(12.21). Physical interpretation follows from an examination of the motion of the normal modes (setting each β_r=1 and other β's=0).

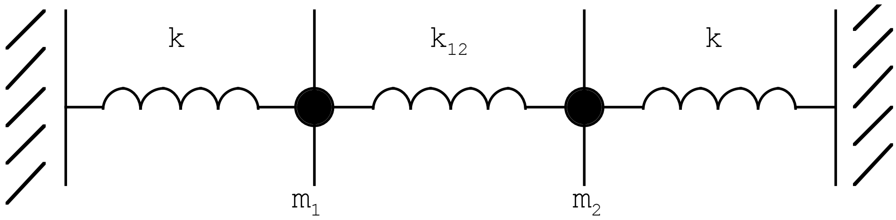
5. Apply the initial conditions and find the β_r.

If T and U for a system are not given, but the equations of motion are, then obviously we may skip step **1** and instead form the characteristic equation using the ansatz Eq.(12.15) for the q_i. As a faster procedure, note that we may also skip the normalization/orthogonality condition, (12.20), and determine $q_j(t) = \sum_r b_j^r e^{i\omega_r t}$ (β_r absorbed in the new b_j^r) directly from the initial conditions.

Note that in step **3**, since the eigenvector equations determine only the ratio of the a_j^r, we need consider only n-1 of the equations in determining all such ratios. Also note that, similar to the case of degenerate eigenvectors for principal axes (see discussion on p.11.16 and prob.#11.8 in the simple case that M=0), there is an indeterminacy in the eigenvectors corresponding to the repeated roots of the characteristic equation. For example, in the case a single repeated root the indeterminacy may be removed by any arbitrary specification of one of the roots – the other will then be determined by Eqs.(12.20).

12.3 EXAMPLE

Enough of the theory, let's do some examples to make this more understandable. Coupled masses example (this is mathematically the same for small oscillations as the coupled pendulum problem at the start of this chapter with $M=m_1=m_2$, $x \to \ell\theta$, and $k = \frac{mg}{\ell}$, ℓ being the pendulum length; see also prob.3.9):



$m_1 = m_2 = M$

Let x_1, x_2 (playing the role of the q's) be measured from *equilibrium* positions. Step ①:

$$T = \frac{1}{2} M \left(\dot{x}_1^2 + \dot{x}_2^2\right).$$

$$\Rightarrow \quad m_{11} = m_{22} = M \quad , \quad m_{12} = m_{21} = 0,$$

$$U = \frac{1}{2} kx_1^2 + \frac{1}{2} kx_2^2 + \frac{1}{2} k_{12}\left(x_2 - x_1\right)^2,$$

$$\Rightarrow A_{11} = \left.\frac{\partial^2 U}{\partial x_1^2}\right|_0 = k + k_{12}, \quad A_{22} = \left.\frac{\partial^2 U}{\partial x_2^2}\right|_0 = k + k_{12},$$

$$A_{12} = \left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_0 = -k_{12}, \quad A_{21} = -k_{12}.$$

Step ②:

$$\det \begin{pmatrix} k + k_{12} - \omega^2 M & -k_{12} \\ -k_{12} & k + k_{12} - \omega^2 M \end{pmatrix} = 0,$$

$$\Rightarrow \left(k + k_{12} - \omega^2 M\right)^2 - k_{12}^2 = 0,$$

$$\Rightarrow \omega_1 = \left(\frac{k + 2k_{12}}{M}\right)^{1/2} \quad , \quad \omega_2 = \left(\frac{k}{M}\right)^{1/2}.$$

Step 3 :

Equations for eigenvectors, r = 1 case (for n=2, only one of the following eigenvector equations is necessary):

$$\begin{cases} \left(A_{11} - \omega_1^2 m_{11}\right) a_1^1 + \left(A_{21} - \omega_1^2 m_{21}\right) a_2^1 = 0, \\ \left(A_{12} - \omega_1^2 m_{12}\right) a_1^1 + \left(A_{22} - \omega_1^2 m_{22}\right) a_2^1 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} -k_{12} a_1^1 - k_{12} a_2^1 = 0 \\ -k_{12} a_1^1 - k_{12} a_2^1 = 0 \end{cases} \Rightarrow \underline{\underline{a_1^1 = -a_2^1}}.$$

r = 2 case:

$$k_{12} a_1^2 - k_{12} a_2^2 \Rightarrow \underline{\underline{a_1^2 = a_2^2}}.$$

Normalization:

$$\sum_{j,k} m_{kj}\, a_j^r a_k^s = \delta_{rs}.$$

overall sign undetermined

$$r = s = 1: \qquad M\left(a_1^1\right)^2 + M\left(a_2^1\right)^2 = 1 \Rightarrow a_1^1 = +\frac{1}{\sqrt{2M}}$$

$$r = s = 2: \qquad M\left(a_1^2\right)^2 + M\left(a_2^2\right)^2 = 1 \Rightarrow a_1^2 = +\frac{1}{\sqrt{2M}}$$

same comment

Step 4:

$$\begin{cases} x_1(t) = \dfrac{1}{\sqrt{2M}} \operatorname{Re}\left(\beta_1 e^{i\omega_1 t} + \beta_2 e^{i\omega_2 t}\right), \\ x_2(t) = \dfrac{1}{\sqrt{2M}} \operatorname{Re}\left(-\beta_1 e^{i\omega_1 t} + \beta_2 e^{i\omega_2 t}\right). \end{cases}$$

[Note we have $(n_r(t) = \beta_r e^{i\omega_r t})$

$$x_1 = \sum_r a_1^r n_r = a_1^1 n_1 + a_1^2 n_2 ,$$

$$\Rightarrow x_1 = \frac{1}{\sqrt{2M}}\left(n_1 + n_2\right).$$

$$x_2 = \sum_r a_2^r n_r = a_2^1 n_1 + a_2^2 n_2 ,$$

$$\Rightarrow x_2 = \frac{1}{\sqrt{2M}}\left(-n_1 + n_2\right).$$

Thus $n_1 = \sqrt{\frac{M}{2}}\left(x_1 - x_2\right), \quad n_2 = \sqrt{\frac{M}{2}}\left(x_1 + x_2\right).$

Physical interpretation of the modes follows from these results. Set $n_1 = 0 \Rightarrow x_1 = x_2$ for mode 2 ("symmetrical mode"). Set $n_2 = 0 \Rightarrow x_1 = -x_2$ ("antisymmetrical mode").]

Step 5 : As an example, consider the initial condition,

$$x_1(0) = -x_2(0) = A, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0.$$

Then

real part of β_1, β_2

$$A = \frac{1}{\sqrt{2M}}\left(\beta_{1r} + \beta_{2r}\right)$$

$$-A = \frac{1}{\sqrt{2M}}\left(-\beta_{1r} + \beta_{2r}\right)$$

$$\Rightarrow \beta_{2r} = 0 \quad , \quad \beta_{1r} = \sqrt{2M}\,A$$

Also

imaginary part

$$\dot{x}_1(0) = -\frac{1}{\sqrt{2M}}\left(\omega_1\beta_{1i} + \omega_2\beta_{2i}\right) = 0,$$

$$\dot{x}_2(0) = \frac{1}{\sqrt{2M}}\left(\omega_1\beta_{1i} - \omega_2\beta_{2i}\right) = 0,$$

$$\Rightarrow \beta_{1i} = \beta_{2i} = 0.$$

If we had chosen $\dot{x}_1(0) = -\dot{x}_2(0) \neq 0$, then we would have had $\beta_{2i} = 0$, $\beta_{1i} = -\dfrac{\sqrt{2M}}{\omega_1}\dot{x}(0)$,

and mode 1 would still be the only one excited. Any combination of the initial conditions $x_1(0) = -x_1(0) = -x_2(0) \neq 0$ and/or $\dot{x}_1(0) = -\dot{x}_2(0) \neq 0$ would have also excited the anti-symmetrical mode alone.

Complete solution for the above boundary conditions:

$$x_1 = A\cos(\omega_1 t), \qquad x_2 = -A\cos(\omega_1 t).$$

The initial conditions $x_1(0) = x_2(0) \neq 0$ and/or $\dot{x}_1(0) = \dot{x}_2(0) \neq 0$ would have excited the symmetrical mode (2) alone. The general motion is a linear combination of the two modes. It is complicated to understand in general, but simplifies in the cases $k_{12} << k$ ("weak coupling") and $k_{12} >> k$ ("strong coupling"). Let's examine these cases in more detail.

12.4 WEAK/STRONG COUPLING

Let's rewrite the above general solution to prepare to discuss weak coupling. Introduce

$$\omega_0 \equiv \frac{\omega_1 + \omega_2}{2} \quad , \quad \underbrace{\omega_b}_{\text{``beats''}} \equiv \frac{\omega_1 - \omega_2}{2} .$$

Can show (mucho algebra!) that the above general solutions for $x_1(t)$, $x_2(t)$ may be written as (solve for ω_1, ω_2 in terms of ω_0, ω_b, substitute above and use trigonometric identities),

$$\begin{aligned} x_1(t) = {} & \frac{A_1}{\sqrt{2M}}\cos(\omega_0 t)\cos(\omega_b t) - \frac{A_2}{\sqrt{2M}}\cos(\omega_0 t)\sin(\omega_b t) \\ & + \frac{A_3}{\sqrt{2M}}\sin(\omega_0 t)\cos(\omega_b t) - \frac{A_4}{\sqrt{2M}}\sin(\omega_0 t)\sin(\omega_b t) , \end{aligned}$$

$$\begin{aligned} x_2(t) = {} & -\frac{A_4}{\sqrt{2M}}\cos(\omega_0 t)\cos(\omega_b t) - \frac{A_3}{\sqrt{2M}}\cos(\omega_0 t)\sin(\omega_b t) \\ & + \frac{A_2}{\sqrt{2M}}\sin(\omega_0 t)\cos(\omega_b t) + \frac{A_1}{\sqrt{2M}}\sin(\omega_0 t)\sin(\omega_b t) . \end{aligned}$$

where

$$A_1 = \beta_{1r} + \beta_{2r} \quad , \quad A_2 = \beta_{1i} - \beta_{2i} ,$$
$$A_3 = -\beta_{1i} - \beta_{2i} \quad , \quad A_4 = \beta_{1r} - \beta_{2r}$$

(Please confirm this.) Notice that there are still only 4 undetermined (real) constants, A_1, A_2, A_3, A_4. Let us now plug in the initial conditions,

$$x_1(0) = D,\ x_2(0) = 0,\ \dot{x}_1(0) = \dot{x}_2(0) = 0.$$

These initial conditions excite both modes 1 and 2, as opposed to the previous set. Plugging in above, we find

$$A_1 = \sqrt{2M}\ D,\ A_2 = A_3 = A_4 = 0.$$

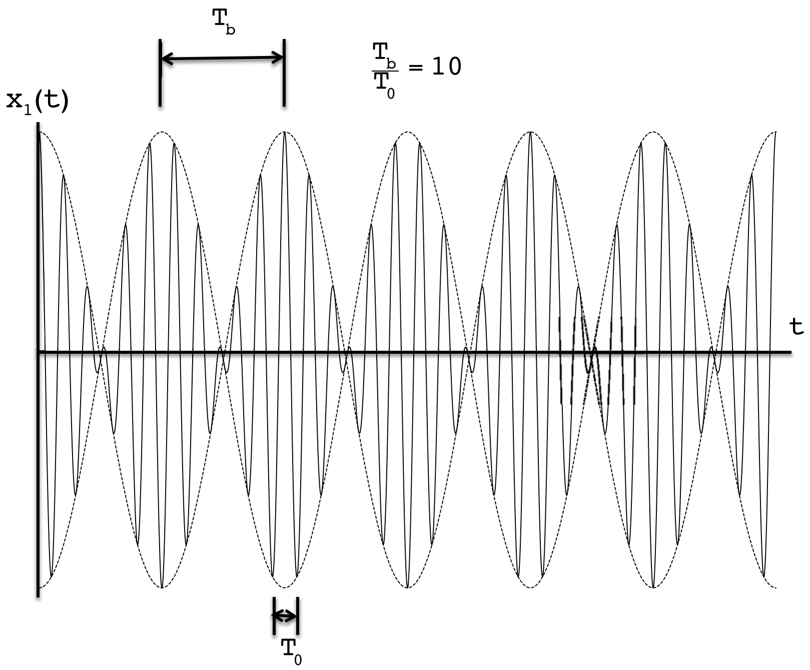
So the general solution is

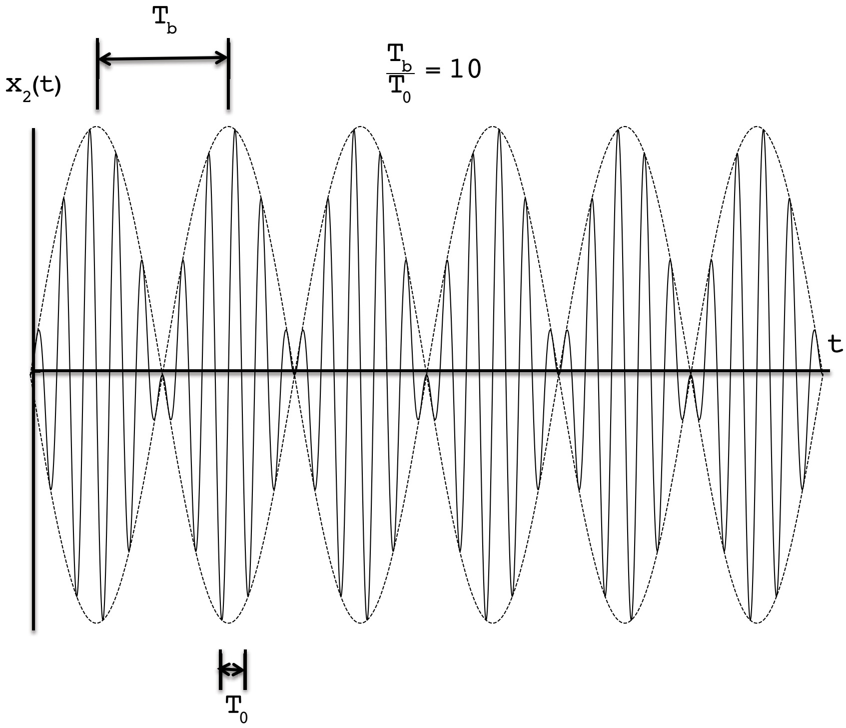
$$x_1(t) = D\cos(\omega_0 t)\cos(\omega_b t) ,$$
$$x_2(t) = D\sin(\omega_0 t)\sin(\omega_b t) .$$

This form is convenient for discussing the case of weak coupling, $k_{12} << k$. Defining $\varepsilon \equiv \dfrac{k_{12}}{2k}$, we have

$$\omega_0 \simeq \sqrt{\frac{k}{M}}\left(1 + \underbrace{\varepsilon}_{\text{small compared to } \omega_0}\right) \quad , \quad \omega_b \simeq \varepsilon\sqrt{\frac{k}{M}} \, .$$

Of course $\sqrt{\dfrac{k}{M}}$ is the decoupled frequency. Motion looks like:





$$T_0 = \frac{2\pi}{\omega_0}\ ,\ T_b = \frac{2\pi}{\omega_b}\ ;\ T_b >> T_0 \text{ in weak coupling.}$$

Phenomenon of *beats* occurs anytime a linear combination of very close frequencies is present. (Listen for them the next time you take a ride on a two-propellar airplane.) This corrresponds to an approximate multiplicative modulation of the uncoupled motion.

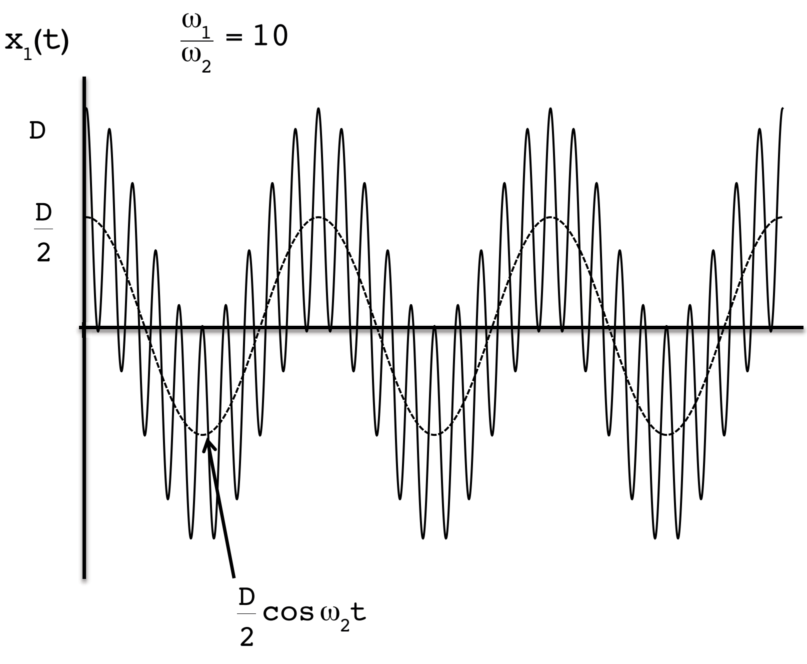
To discuss strong coupling, let us recast the above solution (still exact for the given BC's) as

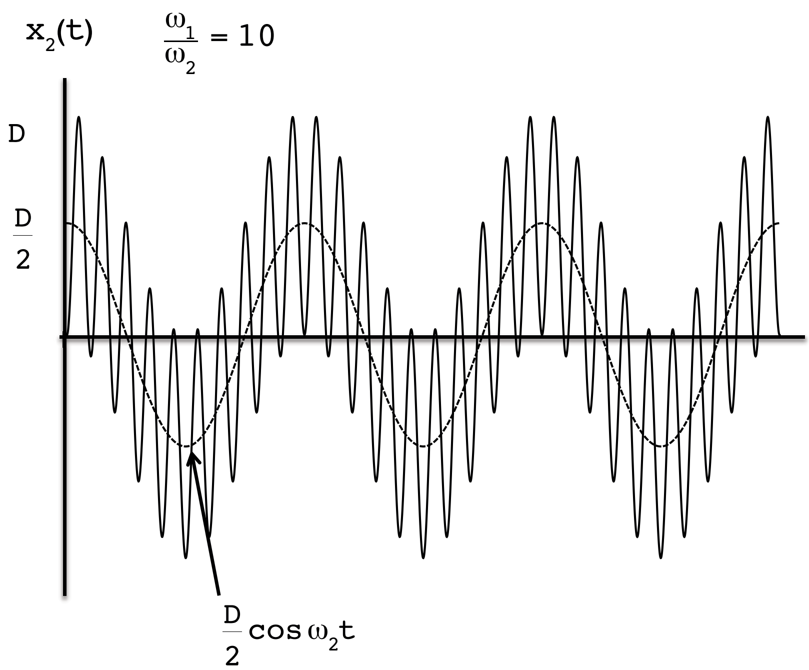
$$\text{also confirm}\quad \begin{cases} x_1(t) = \dfrac{D}{2}\left[\cos\omega_1 t + \cos\omega_2 t\right], \\ x_2(t) = \dfrac{D}{2}\left[-\cos\omega_1 t + \cos\omega_2 t\right]. \end{cases}$$

Then, for strong coupling, $k_{12} >> k$, we have $\left(\tilde{\varepsilon} \equiv \left(\dfrac{k}{2k_{12}}\right)^{1/2}\right)$

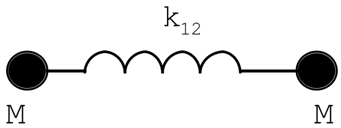
$$\omega_1 \simeq \sqrt{\frac{2k_{12}}{M}}\left(1 + \frac{\tilde{\varepsilon}^2}{2}\right), \quad \omega_2 \simeq \tilde{\varepsilon}\,\omega_1 .$$

We find the motion looks like:

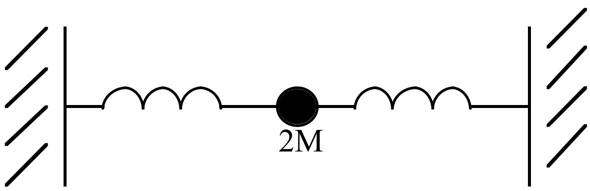




Now get an *additive* modulation. The two frequencies here are understandable from:



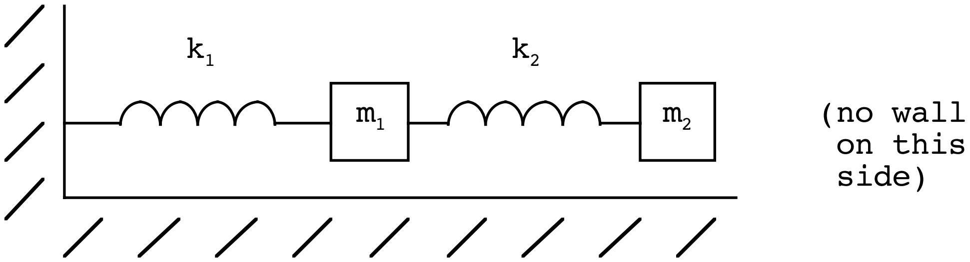
$$\Rightarrow \qquad \omega = \sqrt{\frac{2k_{12}}{M}} \simeq \omega_1 \, ,$$



$$\omega = \sqrt{\frac{2k}{2M}} = \sqrt{\frac{k}{M}} = \omega_2 \, .$$

12.5 EXAMPLE USING MECHANICAL/ELECTRICAL ANALOGY

Let's do one more example. It is a coupled system we encountered before, but we did not solve it because the equations were coupled. We had in Ch.3:



x_1 : length of spring 1

x_2 : length of spring 2

(These lengths are compared to the equilibrium lengths.)

$$m_1\ddot{x}_1 = -k_1x_1 + k_2x_2 \qquad ①$$

$$m_2\left(\ddot{x}_1 + \ddot{x}_2\right) = -k_2x_2 \qquad ②$$

Reminder of the mechanical/electrical analogy:

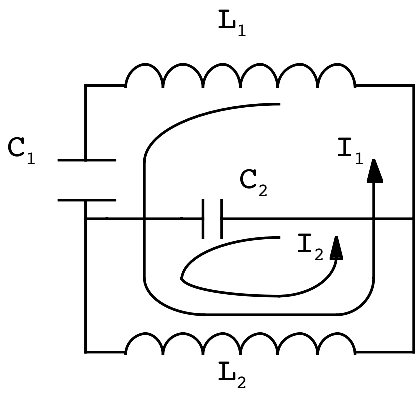
$$F \leftrightarrow V \;,\quad \dot{x} \leftrightarrow I \;,\quad k \leftrightarrow \frac{1}{C}$$

$$x \leftrightarrow q \;,\quad m \leftrightarrow L \;,$$

$$① \quad \Rightarrow \quad L_1\ddot{q}_1 + \frac{1}{C_1}q_1 - \frac{1}{C_2}q_2 = 0\,,$$

$$② \quad \Rightarrow \quad L_2\left(\ddot{q}_1 + \ddot{q}_2\right) + \frac{1}{C_2}q_2 = 0\,.$$

Circuit looks like:



Assume (do not need to know T or U here since we already have the equations of motion)

$$q_{1,2}(t) = a_{1,2}\, e^{i\omega t},$$

$$\begin{cases} \left(\dfrac{1}{C_1} - L_1\omega^2\right)a_1 - \dfrac{1}{C_2}a_2 = 0, \\ -L_2\omega^2 a_1 + \left(\dfrac{1}{C_2} - L_2\omega^2\right)a_2 = 0. \end{cases}$$

$$\Rightarrow \det\begin{pmatrix} \dfrac{1}{C_1} - L_1\omega^2 & -\dfrac{1}{C_2} \\ -L_2\omega^2 & \dfrac{1}{C_2} - L_2\omega^2 \end{pmatrix} = 0\,,$$

$$\Rightarrow \left(\frac{1}{C_1} - L_1\omega^2\right)\left(\frac{1}{C_2} - L_2\omega^2\right) - \frac{1}{C_2}L_2\omega^2 = 0\,,$$

$$\text{or} \qquad L_1L_2\omega^4 - \frac{1}{C_2}L_1\omega^2 - \frac{1}{C_2}L_2\omega^2 - \frac{1}{C_1}L_2\omega^2 + \frac{1}{C_1C_2} = 0\,.$$

A quadratic equation in ω^2. Roots are:

$$\omega^2 = \frac{1}{2}\left[\frac{1}{C_2L_2} + \frac{1}{C_2L_1} + \frac{1}{C_1L_1}\right] \pm \frac{1}{2}\sqrt{\left(\frac{1}{C_2L_2} + \frac{1}{C_2L_1} + \frac{1}{L_1C_1}\right)^2 - \frac{4}{C_1C_2L_1L_2}}\ .$$

Because the above is so complicated, let us just look at a special case: $C_1 = C_2$, $L_1 = L_2$. Then

$$\omega^2 = \frac{1}{LC}\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right).$$

Call

$$\omega_1^2 = \frac{3 + \sqrt{5}}{2LC}\ , \quad \omega_2^2 = \frac{3 - \sqrt{5}}{2LC}\ .$$

These are the normal mode frequencies of this circuit. We will skip the step of normalizing the eigenvectors. The general solution is (real part understood)

$$\left.\begin{aligned} q_1(t) &= a_1^1 e^{i\omega_1 t} + a_1^2 e^{i\omega_2 t} \\ q_2(t) &= a_2^1 e^{i\omega_1 t} + a_2^2 e^{i\omega_2 t} \end{aligned}\right\} \text{8 real constants}$$

where the $\bar{a}^i$ are in general complex. To find the eigenvector relations, substitute ω_1^2, ω_2^2 back into the eigenvector equations:

$$\left(\frac{1}{C} - L\omega_1^2\right)a_1^1 - \frac{1}{C}a_2^1 = 0\ ,$$

$$\Rightarrow a_2^1 = -\left(\frac{1 + \sqrt{5}}{2}\right)a_1^1\ .$$

Likewise

$$\Rightarrow a_2^2 = -\left(\frac{1 - \sqrt{5}}{2}\right)a_1^2\ .$$

We now have,

$$\left.\begin{aligned} q_1(t) &= a_1^1 e^{i\omega_1 t} + a_1^2 e^{i\omega_2 t} \\ q_2(t) &= -\left(\frac{1+\sqrt{5}}{2}\right) a_1^1 e^{i\omega_1 t} - \left(\frac{1-\sqrt{5}}{2}\right) a_1^2 e^{i\omega_2 t} \end{aligned}\right\} \quad \text{4 real constants.}$$

Using $n_{1,2} = e^{i\omega_{1,2} t}$ as the normal mode variables, we find

$$n_1 = \left(\frac{\sqrt{5}-1}{2} q_1 - q_2\right) \frac{1}{\sqrt{5}\, a_1^1},$$

$$n_2 = \left(\frac{1+\sqrt{5}}{2} q_1 + q_2\right) \frac{1}{\sqrt{5}\, a_1^2}.$$

Mode 2 occurs when $n_1 = 0$, so

$$\Rightarrow \quad q_2 = \frac{\sqrt{5}-1}{2} q_1 .$$

Mode 1 occurs when

$$q_2 = -\left(\frac{1+\sqrt{5}}{2}\right) q_1 .$$

These are like the modes for the coupled masses we saw in our first example, except the oscillation amplitudes are unsymmetrical. We can now build in the initial conditions. Let's say

$$\dot{q}_1(0) = \dot{q}_2(0) = 0 \quad , \quad q_1(0) = q_{10}, \; q_2(0) = q_{20} .$$

Plugging in above, we get (remember the a\o($_1$,^{1}), a\o($_1$,^{2}) are complex)

$$0 = \text{Re}\,[i\omega_1 a_1^1 + i\omega_2 a_1^2],$$

$$0 = \text{Re}\left[-\left(\frac{1+\sqrt{5}}{2}\right) i\omega_1 a_1^1 - \left(\frac{1-\sqrt{5}}{2}\right) i\omega_2 a_1^2\right],$$

$$\Rightarrow \quad \underbrace{\left(a_1^1\right)_I = \left(a_1^2\right)_I}_{\text{imaginary part}} = 0 .$$

Likewise

$$q_{10} = \overbrace{\left(a_1^1\right)_R + \left(a_1^2\right)_R}^{\text{real part}},$$

$$q_{20} = -\left(\frac{1+\sqrt{5}}{2}\right)\left(a_1^1\right)_R - \left(\frac{1-\sqrt{5}}{2}\right)\left(a_1^2\right)_R ,$$

$$\begin{cases} \left(a_1^1\right)_R = \dfrac{1}{\sqrt{5}}\left[-q_{20} + \dfrac{\sqrt{5}-1}{2} q_{10}\right], \\ \left(a_1^2\right)_R = \dfrac{1}{\sqrt{5}}\left[q_{20} + \dfrac{1+\sqrt{5}}{2} q_{10}\right]. \end{cases}$$

Full solution:

$$q_1(t) = \frac{1}{\sqrt{5}}\left[-q_{20} + \frac{\sqrt{5}-1}{2} q_{10}\right]\cos\omega_1 t + \frac{1}{\sqrt{5}}\left[q_{20} + \frac{1+\sqrt{5}}{2} q_{10}\right]\cos\omega_2 t ,$$

$$q_2(t) = \frac{1}{\sqrt{5}}\left[-q_{10} + \frac{1+\sqrt{5}}{2} q_{20}\right]\cos\omega_1 t + \frac{1}{\sqrt{5}}\left[q_{10} + \frac{\sqrt{5}-1}{2} q_{20}\right]\cos\omega_2 t .$$

Whew!

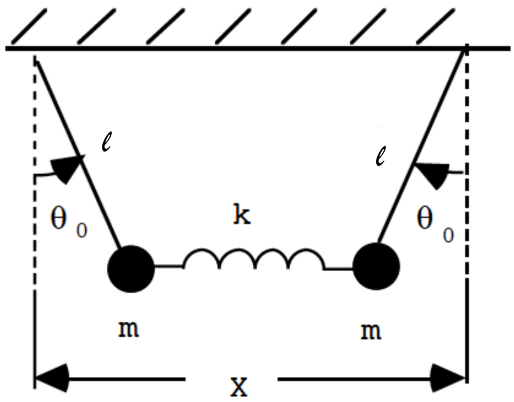
12.6 PROBLEMS

1. Prove that the squared characteristic angular frequencies, ω_r^2, defined from (see Eq.(12.16); no sum on r)

$$\omega_r^2 = \frac{\sum_{i,j} a_i^r A_{ij} a_j^r}{\sum_{i,j} a_i^r m_{ij} a_j^r} ,$$

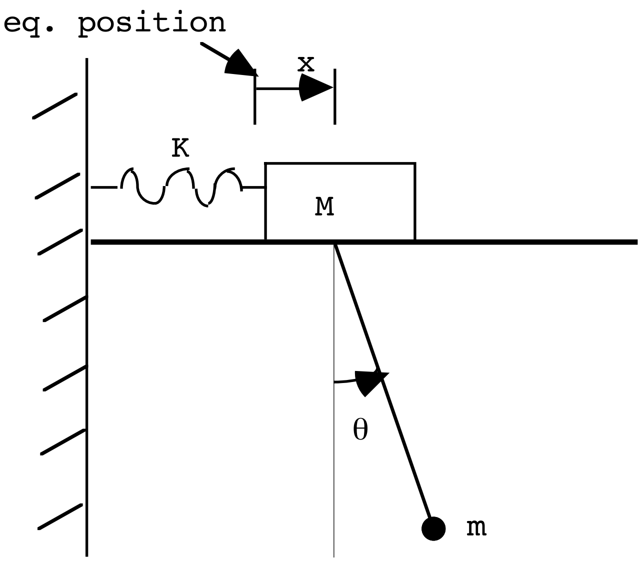
are positive, given the equilibrium conditions in the text.

2. A double pendulum system is arranged such that at equilibrium the pendulums from which the masses, m, are hung are displaced at an angle, θ_0, from vertical as shown. The pendulum lengths are ℓ and the spring constant is k. The unstretched length of the spring is L and X is the distance between the attachment points, as shown.



Find the eigenfrequencies of the system for small oscillations about θ_0. [Hints: First, show that the equilibrium angle, θ_0, is determined by $Y \equiv X - L = 2\ell\sin\theta_0 + \frac{mg}{k}\tan\theta_0$. Then, argue that the potential, U, is given for angles θ_1, θ_2 by $U(\theta_1,\theta_2) = mg\ell(1-\cos(\theta0+\theta1)) + mg\ell(1 - \cos(\theta_0-\theta_2)) + \frac{1}{2}k(Y - \ell\sin(\theta_0+\theta_1) - \ell\sin(\theta_0-\theta_2))^2$. Expand for small angles and solve for the eigenfrequencies.]

3. A mass M moves horizontally along a smooth rail. A spring with spring constant, K, attaches the mass M to the wall. Let x be the distance that the mass M is located from it's equilibrium position.



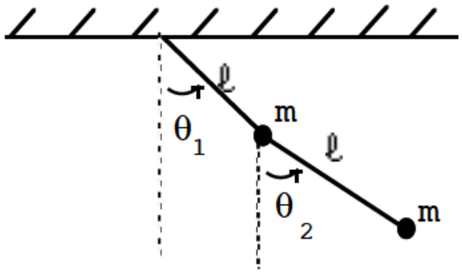
The Lagrangian is (you do not have to derive this)

$$L \approx \frac{1}{2}(M + m)\dot{x}^2 + \frac{m}{2}(b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}) - \frac{mgb}{2}\theta^2 - \frac{K}{2}x^2,$$

in terms of x and θ ($\theta << 1$).

a) Find the squared eigenfrequencies (ω_r^2) of the system.
b) Find the conditions on x and θ which excite each of these modes.

4. In the first semester we considered a double pendulum, consisting of two equal masses connected to each other and a horizontal support



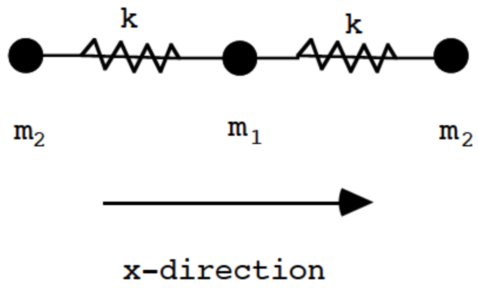
by weightless rods of length ℓ. For small oscillations, the equations of motion we found were,

$$\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2 + \frac{g}{\ell}\theta_1 = 0,$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 + \frac{g}{\ell}\theta_2 = 0.$$

a) Find the characteristic frequencies of the system.
b) Solve for the normal coordinates, n, in terms of θ_1, and θ_2. Describe the conditions which excite these modes.

5. Three masses, arrayed as shown, are coupled together in a straight line with two springs, both with spring constant, k. This is a one-dimensional problem, so motion can only occur along the x-direction.



a) Find the squared eigenfrequencies (ω_r^2) of the system.
b) Find the corresponding eigenvectors (they need not be normalized). Descibe the motion associated with each of the normal modes.

Other Problems

6. In two dimensions, a particle of mass m near the origin experiences the potential,

$$U(x, y) = \frac{1}{2} k_x x^2 + \frac{1}{2} k_y y^2 + k_{xy} xy \,.$$

Given that $k_{xy}^2 < k_x k_y$, and that kx and ky are positive,

a) Find the eigenfrequencies of the system, $\omega_{1,2}^2$.
b) Show that

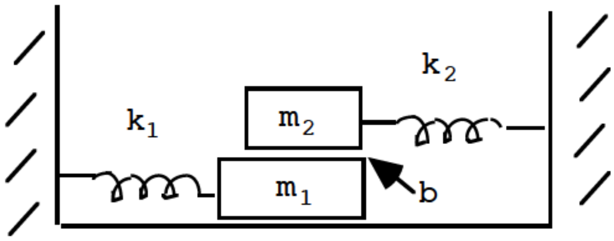
$$y = \frac{x}{2k_{xy}} \left((k_x - k_y) + \sqrt{(k_x - k_y)^2 + 4k_{xy}^2} \right),$$

excites one mode (which one?), and

$$y = \frac{x}{2k_{xy}} \left((k_x - k_y) - \sqrt{(k_x - k_y)^2 + 4k_{xy}^2} \right),$$

excites the other.

7. Prob. 3.8 of this text considers a coupled system of two masses, as shown.



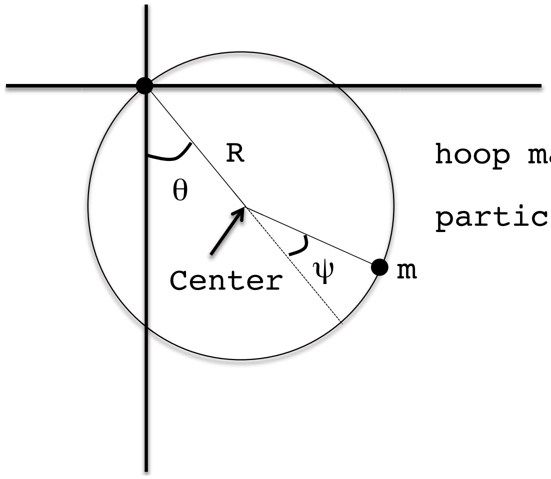
The friction between masses 1 and 2 provides a coupling of the motion proportional to the relative velocity between the masses, $\dot{x}_1 - \dot{x}_2$. Find the equations of motion and discuss coupled oscillations of the system. (There are actually no normal modes here since the friction dissipates the energy of the system. Look for complex characteristic frequencies.)

8. Prove that Eq.(12.19),

$$\sum_{j,k} m_{jk} a_j^r a_k^s = 0 \,, \quad (r \neq s)$$

holds for non-degenerate eigenfrequencies, $\omega_r^2 \neq \omega_s^2$. [Hint: Look at the similar proof in Ch.11.]

9. Consider the hoop-mass system with angles θ and Ψ as shown. Both the particle and hoop have mass m; the radius of the hoop is R.



a) Show that for small oscillations, the Lagrangian is

$$L \approx mR^2(3\dot{\theta}^2 + \frac{1}{2}\dot{\psi}^2 + 2\dot{\theta}\dot{\psi}) - mgR(\frac{3}{2}\theta^2 + \frac{1}{2}\psi^2 + \theta\psi),$$

and the equations of motion are:

$$\frac{g}{a}(\phi + \theta) + (2\ddot{\phi} + \ddot{\theta}) = 0,$$
$$\frac{g}{a}(3\phi + \theta) + (6\ddot{\phi} + 2\ddot{\theta}) = 0.$$

b) Find the normal mode angular frequencies of the system, ω_1 and ω_2.
c) Find the time-independent ratio, $\phi(t)/\theta(t)$, for each of the normal modes of this system.

13 SPECIAL RELATIVITY

13.1 INVARIANCE AND COVARIANCE

Newton's equations are: $(\alpha = 1, ..., n)$

$$m_\alpha \frac{d^2\vec{x}_\alpha}{dt^2} = \vec{f}_\alpha \qquad \text{(3n equations)} \tag{13.1}$$

This is for an n-particle system. Showed under certain circumstances they could be written as

$$\frac{\partial L}{\partial x_{\alpha i}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\alpha i}}\right) = 0, \tag{13.2}$$

where L = T – U. We then have

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{x}}_\alpha^2 , \tag{13.3}$$

$$U = \sum_{\alpha < \beta} U_{\alpha\beta}\left(\left|\vec{x}_\alpha - \vec{x}_\beta\right|\right). \qquad \left(\begin{array}{l} \vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha U_{\alpha\beta}, \\ \vec{f}_\alpha = \sum_{\beta \neq \alpha} \vec{f}_{\alpha\beta}. \end{array}\right) \tag{13.4}$$

Let us try making the transformation,

$$\vec{x}_\alpha \rightarrow \vec{x}_\alpha + \delta\vec{x} . \tag{13.5}$$

in L. Describes a change of origin, a translation. Under this change

$$\left.\begin{array}{l} T \rightarrow T \\ U \rightarrow U \end{array}\right\}$$

Lagrangian is said to be *invariant*. We also learned in Ch.6 that this means something is conserved, in this case linear momentum. On the other hand, consider the transformation

$$\vec{x}_\alpha \rightarrow \vec{x}_\alpha - t\delta\vec{v} , \tag{13.6}$$

in L. Describes a change in velocity or inertial coordinate system. Called a *Galilean transformation* or a *Galilean boost*. Under this transformation,

$$T \rightarrow \frac{1}{2}\sum_{\alpha} m_{\alpha}\left(\dot{\vec{x}}_{\alpha} - \delta\vec{v}\right)^{2},$$

$$U \rightarrow U.$$

Lagrangian is not invariant ⇒ nothing conserved. However, the *equations of motion* derived from the new Lagrangian are unchanged. We still get

$$m_{\alpha}\frac{d^{2}\vec{x}_{\alpha}}{dt^{2}} = \vec{f}_{\alpha},$$

in the boosted frame. The equations themselves are said to be covariant. $\mathtt{Def}^{\underline{n}}\mathtt{s}$:

Invariant: unchanged in *value* as the result of some transformations.

Covariant: Equation or quantity that is unchanged in *form* as the result of a transformation.

13.2 TWO POSTULATES OF SPECIAL RELATIVITY

Physics at the end of the 19th century based upon 3 things:

1. Newton's equations (mechanical phenomena, including gravity and waves.)
2. Maxwell's equations (electromagnetism)
3. Galilean transformations

Although we still accept Maxwell's equations today in the same form as in the 19th century, their interpretation is completely different. Through the influence of the mechanical view of the universe from 1. above and the knowledge that light had wave characteristics (diffraction, interference, etc.), it was assumed that there was a medium for it's propagation, the ether, and that the form of the electromagnetic equations given by Maxwell were valid only for a frame of reference at rest with respect to the ether since the equations were not covariant under Galilean transformations. It was thought that reference frames in uniform motion with respect to each other were completely equivalent as far as mechanical properties were concerned, but not with respect to electromagnetic phenomena. There was only one frame of reference for which, for example, it was thought that the velocity of light had the value predicted by the "rest form" of Maxwell's equations.

The Michelson-Morley experiment (~1887) was designed to test for motion of the Earth relative to the ether. Of course it failed to detect any. So experimentally:

i. The speed of light is an *invariant* under velocity boosts. (That is, it is observed to propagate with the same speed by all inertial observers.)

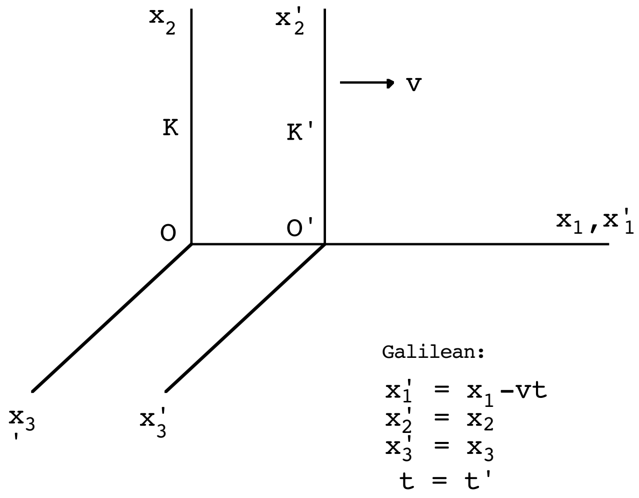
Einstein accepted this fact even though he was, perhaps, not up to date on the M/M experiment. He also realized that the concept of the ether was superfluous. With no ether rest frame, the only frame of reference that can have any significance to an observer is the frame fixed to himself or herself. Therefore, Einstein's second postulate:

ii) The laws of physical phenomena are *covariant* under velocity boosts. (That is, the *form* of the laws are the same for all inertial observers.)

Einstein thought of "physical phenomena" as either electromagnetism or gravity. This has now been enlarged to include other forces; "weak" and "strong" nucleus forces as well as electromagnetism and gravity. (ii) above implies modifying Maxwell's equations or abandoning Galilean transformations. It was the boldness of Einstein which led him to modify the transformation laws between inertial observers.

13.3 LORENTZ TRANFORMATIONS DEDUCED

Consider:



A light pulse is emitted from the common origin of K, K' when they coincide at a single instant of time. According to (i), the wavefront is described by:

$$\text{K:} \quad \bar{r} = c\bar{t} \Rightarrow \bar{r}^2 - c^2\bar{t}^2 = 0, \tag{13.7}$$

$$\text{K}': \quad \bar{r}' = c\bar{t}' \Rightarrow \bar{r}'^2 - c^2\bar{t}'^2 = 0. \tag{13.8}$$

By definition:

$$\bar{r}^2 = \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 = \sum_{i=1}^{3} \bar{x}_i^2, \qquad \bar{r}'^2 = \bar{x}_1'^2 + \bar{x}_2'^2 + \bar{x}_3'^2 = \sum_{i=1}^{3} \bar{x}_i'^2$$

(The bar means coordinates relating to the position of the light pulse.) Or:

$$\text{K:} \quad \sum_i \bar{x}_i^2 - c^2\bar{t}^2 = 0, \tag{13.9}$$

$$\text{K}': \quad \sum_i \bar{x}_i'^2 - c^2 t'^2 = 0. \tag{13.10}$$

Introduce

$$\bar{x}_4 \equiv ic\bar{t} \quad , \quad \bar{x}'_4 \equiv ic\bar{t}'.$$

$$\Rightarrow \quad K: \quad \sum_{\mu=1}^{4} \bar{x}_\mu^{\,2} = 0, \tag{13.11}$$

$$K': \quad \sum_{\mu=1}^{4} \bar{x}'^{\,2}_\mu = 0. \tag{13.12}$$

Usually use i,j,k = 1,2,3 (Latin) but μ, ν, λ = 1,2,3,4 (Greek). $\bar{x}_\mu$ is called a *4-vector*. Because the form of the equation for the light pulse front has the same form in K or K', we realize this relation is covariant under a change of inertial observers. (Although the values of the individual terms change in going from K to K', the equation itself is unchanged in form.) We want the most general (linear) transformation (replacing the Galiliean one) which is consistent with this relation, but which reduces to the known Galilean transformation for low velocities (compared to light). (It is easy to show that $\sum_{\mu=1}^{4} \bar{x}_\mu^{\,2} = 0$ is *not* covariant under the above Galilean transformation.)

Thus, we assume the transformation law has the general form of:

$$\left.\begin{aligned} x_1' &= \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3 + \lambda_{14}x_4\,, \\ x_2' &= \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 + \lambda_{24}x_4\,, \\ x_3' &= \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 + \lambda_{34}x_4\,, \\ x_4' &= \lambda_{41}x_1 + \lambda_{42}x_2 + \lambda_{43}x_3 + \lambda_{44}x_4\,, \end{aligned}\right\} \tag{13.13}$$

where the λ's are not yet known. (Notice we have dropped the bars on the coordinates – we are not necessarily talking about light pulse coordinates at this point, so we do not have $\sum_{\mu=1}^{4} x_\mu^2 = 0$, in general.) The final form of the above will be called a *Lorentz transformation*. Can think of xm, x_μ' as representing the space and time coordinates of an arbitrary space-time "event" from the point of view of two different inertial coordinate systems, the common origin of which is at the place/time where/when the K, K' spacial origins coincide.

The general form of this transformation looks similar to the orthogonal transformations of Ch.1. Actually, orthogonal transformations are consistent with the covariance of $\sum_\mu \bar{x}_\mu^2 = 0$. We had

$$\left\{\begin{aligned} \bar{x}_i' &= \sum_i \lambda_{ij}\bar{x}_j\,, \\ \bar{t}' &= \bar{t}\,. \end{aligned}\right.$$

It preserves the "length" of the space, time terms in the sum individually:

$$\left.\begin{aligned} \sum_i \bar{x}_i'^{\,2} &= \sum_i \bar{x}_i^2\,, \\ \bar{t}'^2 &= \bar{t}^2\,. \end{aligned}\right\} \Rightarrow \sum_\mu \bar{x}_\mu'^{\,2} = 0\,.$$

So, 3-D orthogonal transformations are a specific type of Lorentz transformation. Realizing this, we now think of Lorentz transformations as describing velocity boosts and/or rotations. It also describes more than this, as we will see. (A Poincaire transformation also describes displacements.)

for a L.T.:

$$x_\mu' = \sum_\nu \lambda_{\mu\nu}x_\nu. \quad \left(\begin{array}{c} \text{matrix :} \\ x' = \lambda x \end{array}\right) \tag{13.14}$$

Therefore

$$\sum_\mu x_\mu'^{\,2} = \sum_{\mu,\nu,\gamma} \lambda_{\mu\nu}\lambda_{\mu\gamma}x_\nu x_\gamma\,. \tag{13.15}$$

We want to transform the statement $\sum_\mu \bar{x}_\mu'^2 = 0$ into $\sum_\mu \bar{x}_\mu^2 = 0$. The only way this can be done, given the linear transformation in (13.13) above, is if the $\sum_\mu x_\mu^2$ and $\sum_\mu x_\mu'^2$ quantities are, in general, directly proportional to one another (see also prob. 13.18):

$$\sum_\mu x_\mu'^2 = \overset{\substack{\text{unknown proportionality factor} \\ \downarrow}}{F}\underbrace{\left(\left|\frac{v}{c}\right|\right)}_{\substack{\uparrow \\ \text{no directionality in} \\ \text{the universe (homogeneous} \\ \text{and isotropic)}}} \sum_\mu x_\mu^2 \, . \tag{13.16}$$

Use (ii) above now. Must also have

$$\sum_\mu x_\mu^2 = F\left(\left|\frac{v}{c}\right|\right)\sum_\mu x_\mu'^2 ,$$

$$\Rightarrow \quad \sum_\mu x_\mu'^2 = F^2\left(\left|\frac{v}{c}\right|\right)\sum_\mu x_\mu'^2 ,$$

$$\Rightarrow \quad F = \pm 1.$$

The minus sign (F = -1) is eliminated for two reasons:

1. The transformation must be continuously connected to the identity transformation, $x_\mu' = x_\mu$.
2. Causality of events (seen later).

Therefore $\sum_\mu x_\mu'^2 = \sum_\mu x_\mu^2$ and so

$$\sum_{\mu,\nu,\gamma} \lambda_{\mu\nu}\lambda_{\mu\gamma}x_\nu x_\gamma = \sum_\mu x_\mu^2 \, . \tag{13.17}$$

Coefficients of individual $x_\nu x_\gamma$ terms must be equal:

$$\Rightarrow \sum_\mu \lambda_{\mu\nu}\lambda_{\mu\gamma} = \delta_{\nu\gamma} \, . \tag{13.18}$$

Reminds us of (Ch.1)

$$\sum_{\substack{i \\ \uparrow \\ \text{notice}}} \lambda_{ij}\lambda_{ik} = \delta_{jk}.$$

We then wrote this as

$$\lambda^{T}\,\lambda = 1.$$

But

$$\lambda^{-1}\,\lambda = 1 \Rightarrow \lambda^{T} = \lambda^{-1} \text{ (defines an orthogonal transf.)}$$

Then

$$\lambda\lambda^{-1} = 1 \Rightarrow \lambda\lambda^{T} = 1,$$

$$\Rightarrow \sum_{i} \underset{\substack{\uparrow}}{\lambda_{ji}}\ \underset{\substack{\uparrow}}{\lambda_{ki}} = \delta_{jk}.$$

notice

Situation is essentially identical here, except λ's are now 4-D quantities:

$$\sum_{\mu} \lambda_{\mu\nu}\lambda_{\mu\gamma} = \delta_{\nu\gamma} \Rightarrow \lambda^{T}\lambda = 1. \tag{13.19}$$

Same chain of reasoning leads to

$$\lambda^{T} = \lambda^{-1} \text{ and } \sum_{\mu} \lambda_{\nu\mu}\lambda_{\gamma\mu} = \delta_{\nu\gamma}. \tag{13.20}$$

Go back to 3-D again. An explicit rotation about the 3 axis is given by

$$\left.\begin{aligned} x_1' &= x_1 \cos\theta + x_2 \sin\theta, \\ x_2' &= -x_1 \sin\theta + x_2 \cos\theta, \\ x_3' &= x_3, \ (t' = t \text{ also, of course}) \end{aligned}\right\} \tag{13.21}$$

From which the λ_{ij} were identified as the *direction cosines* of the rotation. The same form must hold in 4-D, except that when we make the physical identification, x_4 = ict, some of the "rotation" parameters will turn out to be imaginary. So, "rotate" in the 1,4 plane:

$$\left\{\begin{aligned} x_1' &= x_1 \cos\psi + x_4 \sin\psi, \\ x_2' &= x_2, \\ x_3' &= x_3, \\ x_4' &= -x_1 \sin\psi + x_4 \cos\psi. \end{aligned}\right.$$

Tie this in with physics now. For the origin of K', $x_1' = 0$, we must have x_1 = vt:

$$0 = vt \cos\psi + \underset{\substack{\uparrow \\ ict}}{x_4} \sin\psi$$

$$\Rightarrow v = -ic \tan\psi \quad \text{or} \quad \tan\psi = i\frac{v}{c}.$$

Since v is real $\Rightarrow$ ψ purely imaginary. Let $\psi = i\alpha \quad (\alpha \text{ real})$

$$\tan\psi = \tan(i\alpha) = i\tanh\alpha, \quad \left(\Rightarrow \tanh\alpha = \frac{v}{c}\right)$$
$$\sin\psi = \sin(i\alpha) = i\sinh\alpha,$$
$$\cos\psi = \cos(i\alpha) = \cosh(\alpha).$$

So, in terms of real parameters, this "rotation" is

$$\begin{cases} x_1' = x_1\cosh\alpha - ct\sinh\alpha, \\ x_2' = x_2, \\ x_3' = x_3, \\ ict' = -ix_1\sinh\alpha + ict\cosh\alpha \end{cases}$$

Then

$$\sin^2\psi = \frac{1}{1+\cot^2\psi} = \frac{1}{1-\frac{c^2}{v^2}} = \frac{\frac{v^2}{c^2}}{\frac{v^2}{c^2}-1},$$

$$\text{``} \quad = -\sinh^2\alpha$$

$$\cos^2\psi = \frac{1}{1+\tan^2\psi} = \frac{1}{1-\frac{v^2}{c^2}},$$

$$\text{``} \quad = \cosh^2\alpha$$

From the above, sin y must be imaginary $\Rightarrow$ $\frac{v}{c} < 1$.

$$\Rightarrow \sinh\alpha = \frac{\pm\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \cosh\alpha = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

We want this to reduce to

$$\begin{cases} x_1' = x_1 - vt\,, \\ x_2' = x_2, \\ x_3' = x_3, \\ t' = t \end{cases}$$

when $\dfrac{v}{c} << 1$. Therefore

$$\sinh\alpha = \frac{+\dfrac{v}{c}}{\sqrt{1 - \dfrac{v^2}{c^2}}}\,, \qquad \cosh\alpha = \frac{+1}{\sqrt{1 - \dfrac{v^2}{c^2}}}\,.$$

So we have

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 - vt\right), \\ x_2' &= x_2\,, \\ x_3' &= x_3\,, \\ t' &= \gamma\left(t - x_1\frac{v}{c^2}\right), \end{aligned}\right\} \begin{array}{c} \text{A Lorentz } \underline{\text{boost}} \\ \gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}\,,\ \beta \equiv \dfrac{v}{c}\,. \end{array} \tag{13.22}$$

The *form* of this transformation was known before Einstein by Lorentz, although it was Einstein who gave it the correct physical interpretation. There is a lot more to physics than just getting the equations correct!

This transformation was based upon requiring the invariance of

$$\sum_\mu \bar{x}_\mu^2 = 0\,.$$

This in turn implied

$$\sum_\mu x_\mu^2 = \sum_\mu x_\mu'^2 \quad (\neq 0)\,.$$

This means $\sum_\mu x_\mu^2$ *is* an *invariant* under a Lorentz transformation. Let's call it "invariant squared distance" and give it a special symbol:

$$\underset{\substack{\uparrow \\ \text{(can be + or −)}}}{s^2} \equiv \sum_\mu x_\mu^2 \tag{13.23}$$

There is no reference to the coordinates of a light pulse here; it is a completely general relation relating lengths and times in arbitrary inertial reference systems. It actually relates the length and time *intervals* between two arbitrary space-time events if we write

$$\Delta s^2 \equiv \sum_\mu \Delta x_\mu^2\,. \tag{13.24}$$

Then the first event is no longer required to happen at the common origin of the K, K' systems. So, we have assumed one invariant, the speed of light,

$$c = c'\,, \tag{13.25}$$

and derived another, the squared invariant distance,

$$\Delta s^2 = \Delta s'^2 \ . \tag{13.26}$$

Thus, everything is *not* relative in relavity; there are still *absolute* quantities.

13.4 ALTERNATE NOTATION FOR LORENTZ TRANSFORMATIONS

One often sees an alternate treatment of Lorentz transformations with no reference to imaginary numbers. It is often used in particle physics. Define[2]

$$x^0 \equiv ct. \quad \text{(no i!)} \tag{13.27}$$

Then a Lorentz transformation is given as:

notice: upper index.

$$\overset{\downarrow}{x'^{\mu}} = \sum_{\nu} \underset{\uparrow}{\Lambda^{\mu}{}_{\nu}} x^{\nu}. \qquad (\mu, \nu = 0, 1, 2, 3) \tag{13.28}$$

different from previous λ's
(they are real)

We identify from above,

$$\Lambda^i{}_j = \lambda_{ij}, \Lambda^0{}_i = -i\lambda_{4i}, \Lambda^i{}_0 = i\lambda_{i4}, \Lambda^0{}_0 = \lambda_{44}.$$

Now

all important minus sign missing!

$$\sum_{\mu} \left(x^{\mu}\right)^2 = \sum_{i} \left(x^i\right)^2 \overset{\downarrow}{+} c^2t^2 \ .$$

Not an invariant! Define ("metric tensor")

$$g \equiv \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array} \qquad \left(g_{\mu\nu} = g_{\nu\mu}\right) \tag{13.29}$$

$g_{\mu\nu}$: $g_{11} = g_{22} = g_{33} = 1$, $g_{00} = -1$, all other components = 0.

Then

$$\sum_{\mu,\nu} x^{\mu} g_{\mu\nu} x^{\nu} = \sum_{i} \left(x^{i}\right)^{2} \overset{\substack{\text{notice}\\ \downarrow}}{-} c^{2}t^{2} \,. \tag{13.30}$$

This *is* an invariant. Sometimes define

$$x_{\mu} = \sum_{\nu} g_{\mu\nu} x^{\nu} \,. \tag{13.31}$$

Notice:

$$x^{\mu} = (ct, \bar{x}), \quad \text{"contravariant 4-vector"}$$
$$x_{\mu} = (-ct, \bar{x}), \quad \text{"covariant 4-vector".}$$

Have to be careful in this notation to recognize whether an index is up or down. Then we may write

$$s^{2} = \sum_{\mu,\nu} x'^{\mu} g_{\mu\nu} x'^{\nu} = \sum_{\mu,\nu} x^{\mu} g_{\mu\nu} x^{\nu} \,. \tag{13.32}$$

or, using the transformation law for x'^{μ}:

$$\Rightarrow \sum_{\mu,\nu,\gamma,\kappa} g_{\mu\nu}\, \Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\kappa} x^{\lambda} x^{\kappa} = \sum_{\gamma,\kappa} g_{\lambda\kappa} x^{\lambda} x^{\kappa},$$

$$\Rightarrow \sum_{\mu,\nu} g_{\mu\nu}\, \Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\kappa} = g_{\lambda\kappa}. \tag{13.33}$$

This is the analog, in this new notation, of the equations satisfied by the $\lambda_{\mu\nu}$'s (see Eqs.(13.20) above) and can be considered an alternate definition of a Lorentz transformation.

Einstein summation convention: sum on any repeated upper, lower indices. (As we saw earlier a repeated index like $x^{\nu}x^{\nu}$ does *not* produce an invariant quantity; this property generalizes.)

The above may now be written:

$$g_{\mu\nu}\Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\kappa} = g_{\lambda\kappa} \tag{13.34}$$

There is also a matrix interpretation of this form. Write (the "T" means transpose; notice the matrix version can violate the Einstein summation convention)

$$\left(\Lambda^{T}\right)^{\lambda}{}_{\mu} g_{\mu\nu}\, \Lambda^{\nu}{}_{\kappa} = g_{\lambda\kappa}, \tag{13.35}$$

$$\Rightarrow \Lambda^{T} g \Lambda = g. \text{ (Takes the place of } \lambda^{T}\lambda = 1.) \tag{13.36}$$

No matter which notation and or conventions we use, it is important to realize the point of either is to build in the minus sign in front of the time components. We will primarily use the Λ^{μ}_{ν} –type notation in which all quantities are real. Famous quote: The start of any calculation is to check the author's conventions.

In order to be able to use this notation, must know the rules. Just as for 3-D rotations, 4-vectors are defined by their transformation laws. We have

$$A'^{\mu} = \sum_{\lambda} \Lambda^{\mu}{}_{\lambda} A^{\lambda} \text{ or } = \Lambda^{\mu}{}_{\lambda} A^{\lambda}, \tag{13.37}$$

where $\Lambda^{\mu}{}_{\lambda}$ satisfies $\Lambda^{T} g \Lambda = g$. This makes $A\mu$ a contravariant 4-vector. To make a covariant 4-vector out of A^{μ}, say, just "contract" with $g_{\mu\nu}$:

$$A_{\mu} \equiv g_{\mu\nu} A^{\nu}. \tag{13.38}$$

Define:

new quantity

$$\underset{\substack{\uparrow \\ \text{Kronecker delta}}}{\overset{\substack{\text{new quantity} \\ \downarrow}}{g^{\mu\nu}}} g_{\nu\kappa} \equiv \delta^{\mu}_{\kappa}, \tag{13.39}$$

Kronecker delta

$$\Rightarrow A^{\mu} \equiv g^{\mu\nu} A_{\nu}. \tag{13.40}$$

Note from (13.39) that the $g^{\mu\nu}$ are just the inverses of the $g_{\mu\nu}$, considered as matrix elements: $g^{\mu\nu} g_{\nu\kappa} = \delta^{\mu}_{\kappa} \Rightarrow g^{-1} g = 1$. The matrix representation of the $g^{\mu\nu}$ in this case is also given by (13.29), which is the so-called flat space metric.

Define $\Lambda_{\mu}^{\ \lambda} \equiv g_{\mu\nu} g^{\lambda\kappa} \Lambda^{\nu}_{\ \kappa}$; then the transformation law for covariant 4-vectors is just:

$$A'_{\mu} = g_{\mu\kappa} A'^{\kappa} = g_{\mu\kappa} \Lambda^{\kappa}_{\ \lambda} A^{\lambda} = \overbrace{g_{\mu\kappa} g^{\nu\lambda} \Lambda^{\kappa}_{\lambda}}^{\Lambda_{\mu}^{\ \nu}} A_{\nu},$$

$$\Rightarrow A'_{\mu} = \Lambda^{\nu}_{\mu} A_{\nu}. \tag{13.41}$$

The matrix interpretation of these transformation laws is based upon the index to matrix correspondence,

$$\Lambda^{\mu}_{\ \nu} \rightarrow (\Lambda)_{\mu\nu} \Rightarrow \Lambda_{\lambda}^{\ \kappa} \rightarrow [(\Lambda^{-1})^{T}]_{\lambda\kappa}.$$

(The second statement follows from the definition $\Lambda_{\mu}^{\ \lambda} \equiv g_{\mu\nu} g^{\lambda\kappa} \Lambda^{\nu}_{\ \kappa}$.) Notice the matrix interpretation ignores the "up" or "down" position of the indices; I have placed these arbitrarily down. This implies the equivalence of the statements,

$$A'^{\mu} = \Lambda^{\mu}_{\ \nu} A^{\nu} \quad \Leftrightarrow \quad A' = \Lambda A, \ (A' \text{ a column matix})$$

$$A'_{\mu} = \Lambda_{\mu}^{\ \nu} A_{\nu} \quad \Leftrightarrow \quad A'^{T} = A^{T} (\Lambda)^{-1}. \ (A'^{T} \text{ a row matrix})$$

Likewise (see prob. 13.2(a))

$$\Lambda^{\nu}_{\ \lambda} \Lambda_{\mu}^{\ \lambda} = \delta^{\nu}_{\mu} \quad \Leftrightarrow \quad \Lambda\left(\left(\Lambda^{-1}\right)^{T}\right)^{T} = 1 \text{ or } \Lambda\Lambda^{-1} = 1.$$

A way to form a Lorentz invariant quantity, or in short, a *scalar*, is to form (just like we did for x^{μ}, x_{μ}),

$$A'^{\mu}A'_{\mu} = A'^{\mu}g_{\mu\nu}A'^{\nu} = \underbrace{g_{\mu\nu}\Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\kappa}}_{g_{\lambda\kappa}} A^{\lambda}A^{\kappa}$$

$$\Rightarrow A'^{\mu}A'_{\mu} = A^{\mu}A_{\mu}\,. \qquad (13.42)$$

Of course $s^2 = x^{\mu}x_{\mu}$ is also a scalar. We have the first of our "index jockey" rules:

1. To make a Lorentz scalar out of two vectors, sum on one upper, one lower index.
   We know that the components of the gradient operator,

$$\nabla_i \equiv \frac{\partial}{\partial x^i}, \qquad (13.43)$$

transform as a vector. Can then build vectors ($\vec{\nabla}\phi$, for ϕ a scalar) or scalars ($\vec{\nabla}\times\vec{A}$ for $\vec{A}$ a 3-vector) for $\vec{A}$ a 3-vector) out of it. For Lorentz transformations, define

$$\nabla_\mu \equiv \frac{\partial}{\partial x^\mu} \quad , \quad \nabla_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right). \tag{13.44}$$

$\uparrow$ lower $\quad$ $\uparrow$ upper

Can show it transforms as a covariant 4-vector (thus, the *lower* index). Also

$$\nabla^\mu \equiv \frac{\partial}{\partial x_\mu}, \nabla^\mu = g^{\mu\nu}\nabla_\nu \Rightarrow \nabla^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right). \tag{13.45}$$

$\uparrow$ upper $\quad$ $\uparrow$ lower

transforms as a contravariant 4-vector. So, the second index jockey rule is:

2. ∇^μ transforms as a contravariant 4-vector
   ∇^μ transforms as a covariant 4-vector.

Two other rules are:

3. Lower an index with $g_{\mu\nu}$: $g_{\mu\nu}A^\nu = A_\mu$.
4. Raise an index with $g^{\mu\nu}$: $g^{\mu\nu}A_\nu = A^\mu$.

As before: $g^{\mu\nu} \; g_{\nu\kappa} = \delta^\mu_\kappa$. $\left(\begin{array}{c} \delta^0_0 = \delta^1_1 = \delta^2_2 = \delta^3_3 = 1, \\ \text{all others} = 0 \end{array}\right)$

Matrix statement: $\quad (g^{-1})g = 1.$

$$g_{\mu\nu} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array} \qquad (\Rightarrow \nabla^\mu = g^{\mu\nu}\nabla_\nu)$$

Note:

* Using $g_{\mu\nu}$, $g^{\mu\nu}$ raises or lowers an index but never changes index *order*.
* Number of *free* up, down indices on one side of an equation must equal the number on other side.

Examples:

1. Show $A^\mu B_\mu = A_\mu B^\mu$: $A^\mu B_\mu = A_\sigma g^{\sigma\mu} B^\tau g_{\tau\mu} = \delta^\sigma_\tau A_\sigma B^\tau = A_\mu B^\mu$. (In general $A^{\cdots\mu\cdots}B_{\cdots\mu\cdots} = A_{\cdots\mu\cdots} B^{\cdots\mu\cdots}$ for arbitrary tensors $A^{\cdots\mu\cdots}$ and $B_{\cdots\mu\cdots}$.)

2. Given $A'^\mu = \Lambda^\mu_\nu A^\nu$, $\tau' = \tau$, prove that $\frac{dA'^\mu}{d\tau'}$ transforms as a contravariant 4-vector:

$$\frac{dA'^\mu}{d\tau'} = \Lambda^\mu_\nu \frac{dA^\nu}{d\tau'} = \Lambda^\mu_\nu \frac{dA^\mu}{d\tau}\underbrace{\frac{d\tau}{d\tau'}}_{1} = \Lambda^\mu_\nu \frac{dA^\mu}{d\tau}.$$

3. Prove that $\nabla'_\mu\phi'$, $\phi' = \phi$, transforms as a covariant 4-vector:

$$\nabla'_\mu\phi' = \frac{\partial\phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\phi}{\partial x^\nu} = \overset{\substack{\text{need}\\ \downarrow}}{\frac{\partial x^\nu}{\partial x'^\mu}}\nabla_\nu\phi .$$

But

$$\left(x'^\lambda = \Lambda^\lambda{}_\kappa x^\kappa\right)\cdot g_{\lambda\alpha}\Lambda^\alpha{}_\beta ,$$

$$\Rightarrow \left(g_{\lambda\alpha}\Lambda^\alpha{}_\beta x'^\lambda = g_{\kappa\beta}x^\kappa\right)\cdot g^{\beta\gamma},$$

$$\Rightarrow \underbrace{g^{\beta\gamma}g_{\lambda\alpha}\Lambda^\alpha{}_\beta}_{\Lambda_\lambda{}^\gamma} x'^\lambda = x^\gamma ,$$

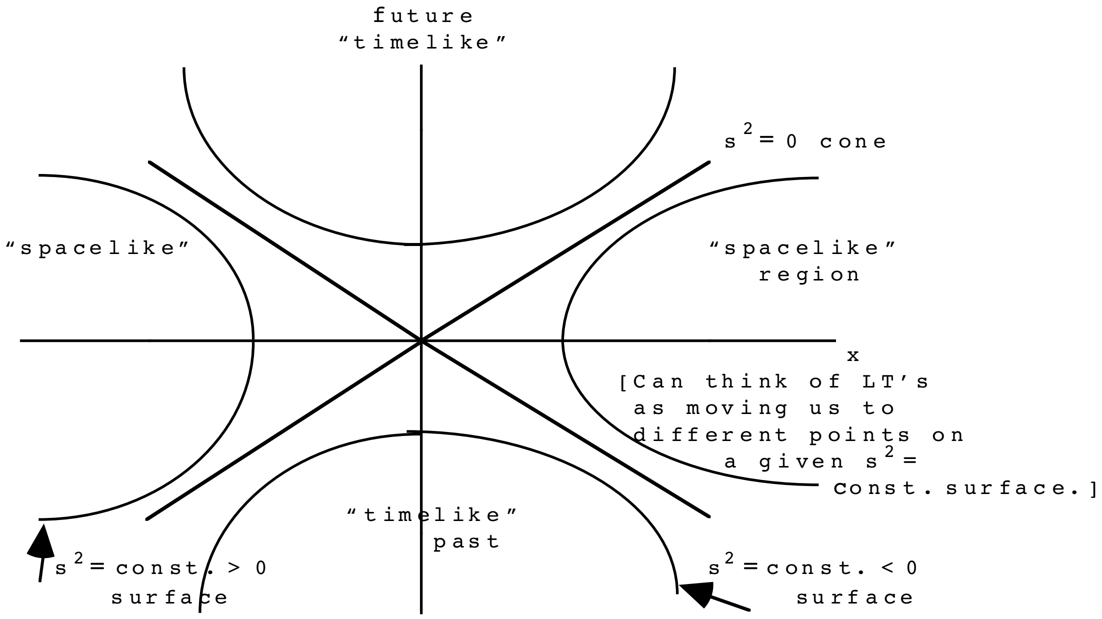
$$\Rightarrow \frac{\partial x^\gamma}{\partial x'^\lambda} = \Lambda_\lambda{}^\gamma . \text{ (Also } \frac{\partial x'^\lambda}{\partial x^\kappa} = \Lambda^\lambda{}_\kappa \text{)}$$

Therefore

$$\nabla'_\mu\phi' = \Lambda_\mu{}^\nu\nabla_\nu\phi \Rightarrow \text{covariant 4-vector.}$$

13.5 THE "LIGHT CONE" AND TACHYONS

Let's get back to the physics being described. For this, we need a picture of the *light-cone* (imagine rotating about the time axis; the "surfaces" shown are hyperboloids):



"timelike" $s^2 < 0$ means $|c\Delta t| > |\Delta \vec{x}|$,

"spacelike" $s^2 > 0$ means $|c\Delta t| < |\Delta \vec{x}|$,

"lightlike: $s^2 = 0$ means $|c\Delta t| = |\Delta \vec{x}|$.

This diagram describes the relationship of the origin to all other space/time points. Events with a *timelike* behavior can always be given a definite *causal* order since a light beam can reach the second event before it occurs. Also, $|c\Delta t| > |\Delta \vec{x}|$ is preserved by Lorentz transformations. This means that spacelike and timelike type events are not mixed. On the other hand, *spacelike* events are fundamentally different: there is no invariant order of these events seen by all observers. Thus:

Timelike events have an invariant causal order
Spacelike events do not have an invariant causal order

Better not mix up such distinct types of events! This is one reason for discarding the Lorentz transformation with F = -1, considered earlier, because such a transformation on the light cone would take us from a timelike ordering of events to a spacelike one, or vice versa. Mathematically, this means we are prevented from jumping across the lightcone.

To see this transformation more explicitly, consider (the "rotation" interpretation of this 1,4 transformation no longer holds because of the factors of "i"):

$$x_1' = i x_1 \sin\psi - i x_4 \cos\psi , \tag{13.46}$$

$$x_4' = i x_1 \cos\psi + i x_4 \sin\psi . \tag{13.47}$$

If we insist as before that the origin of K', $x_1' = 0$, has x_1, has then $= vt$, then $\tan\psi = i\frac{c}{v}$, and

$$\sin^2\psi = \frac{1}{1 - \frac{v^2}{c^2}} . \tag{13.48}$$

$\sin\psi$ must be chosen imaginary:

$$\Rightarrow \frac{v^2}{c^2} > 1 . \tag{13.49}$$

Hypothetical particles with $\frac{v}{c} > 1$ are called *tachyons*. These particles can be shown to move either backward or forward in time, depending on the direction of motion! So by eliminating F = -1, we are eliminating the possibility of transformations to frames of reference with $\frac{v}{c} > 1$.

13.6 MATHEMATICAL PROPERTIES OF LORENTZ TRANSFORMATIONS

Let's do a little investigation of mathematical properties of Lorentz transformations. Simple facts about determinants:

$$\det AB = \det A \ \det B \quad , \quad \det A^T = \det A.$$

Since

$$\Lambda^T g \Lambda = g\,, \tag{13.50}$$

taking determinant of both sides gives,

$$\det\left[\Lambda^T g \Lambda\right] = (\det \Lambda)^2 \det g = \det g\,,$$

$$\Rightarrow \quad \det \Lambda = \pm 1. \tag{13.51}$$

+1: ``Proper'' Lorentz transformation

-1: ``Improper'' Lorentz transformation

Will see the physical interpretation later. Also, we have

$$x' = \Lambda_1 x, \tag{13.52}$$

$$x'' = \Lambda_2 x', \tag{13.53}$$

$$x'' = \Lambda_2 \Lambda_1 x \equiv \Lambda_{21} x. \tag{13.54}$$

Is Λ_{21} also a Lorentz transformation? Use the definition:

$$\Lambda_2^T g \Lambda_2 = g\,, \tag{13.55}$$

$$\Rightarrow \Lambda_1^T \Lambda_2^T g \Lambda_2 \Lambda_1 = \Lambda_1^T g \Lambda_1 = g\,, \tag{13.56}$$

However, $\Lambda_{21}^{T} = \left(\Lambda_2\Lambda_1\right)^{T} = \Lambda_1^{T}\Lambda_2^{T}$, so

$$\Rightarrow \Lambda_{21}^{T}\, g\, \Lambda_{21} = g\,. \tag{13.57}$$

$\Rightarrow \Lambda_{21}$ is also a Lorentz transformation

How many independent elements of $\Lambda^{\mu}{}_{\nu}$ are there? Expect physically:

$$3\ (\text{rotations}) + 3\ (\text{boosts}) = \underline{\underline{6}}\,.$$

Mathematically: Λ^{μ}_{ν}, $4 \times 4 = 16$, real quantities. However, the equation

$$\Lambda^{T} g \Lambda = g\,,$$

is also (like 3-D rotation) unchanged by the transpose. Therefore, there are as many components as in a symmetric 4 × 4 matrix:

$$\begin{pmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{pmatrix} \qquad \underline{\underline{10}}$$

This means there are 16 – 10 = 6 independent real quantities, the right number.

Some terminology:

Restricted Lorentz Transformation (RLT): Transformations physically possible for a frame of reference associated with a material observer.

What this means is rotation + boosts (I will be more precise mathematically in a bit.) We have learned that there are Lorentz transformations with det Λ = +1 or -1. There is a further discontinuous possibility as follows. Take the 00 component of the defining equation:

$$g_{\mu\nu}\Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\kappa} = g_{\lambda\kappa} \qquad , \qquad \lambda = \kappa = 0\,.$$

$$\Rightarrow g_{00}\Lambda^{0}{}_{0}\Lambda^{0}{}_{0} + g_{11}\Lambda^{1}{}_{0}\Lambda^{1}{}_{0} + g_{22}\Lambda^{2}{}_{0}\Lambda^{2}{}_{0} + g_{33}\Lambda^{3}{}_{0}\Lambda^{3}{}_{0} = g_{00}\,, \tag{13.58}$$

$$g_{00} = -1, \quad g_{ii} = 1,$$

$$\Rightarrow -\left(\Lambda^0{}_0\right)^2 + \sum_i \left(\Lambda^i{}_0\right)^2 = -1, \tag{13.59}$$

$$\Rightarrow \left(\Lambda^0{}_0\right)^2 = 1 + \sum_i \left(\Lambda^i{}_0\right)^2 \geq 1. \quad \left(\begin{array}{c} \text{Can also show :} \\ \left(\Lambda^0{}_0\right)^2 = 1 + \sum_i \left(\Lambda^0{}_i\right)^2 . \end{array} \right) \tag{13.60}$$

So, there are two possibilities for $\Lambda^0{}_0$:

$$\Lambda^0{}_0 \geq 1 \qquad , \qquad \Lambda^0{}_0 \leq -1$$

"Orthochronous" "Nonorthochronous"

Now our more precise mathematical definition of a restricted Lorentz transformation:

RLT:

Proper, orthochronous Lorentz transformation:

$$\det \Lambda = 1, \ \Lambda^0{}_0 \geq 1, \ \Lambda^T g \Lambda = g .$$

Prove: If Λ_1 and Λ_2 are both RLT's then so is $\Lambda_{21} = \Lambda_2 \Lambda_1$.

Show $\det \Lambda_{21} = 1$ (easy):

$$\det \Lambda_{21} = \det \Lambda_2 \det \Lambda_1 = 1.$$

Show that $(\Lambda_{21})^0{}_0 \equiv (\Lambda_2)^0{}_\gamma (\Lambda_1)^\gamma{}_0 \geq 1$ (not so easy):From the above

$$\Lambda_{21} = \Lambda_2\Lambda_1 \qquad \text{or} \qquad (\Lambda_{21})^\mu{}_\nu = (\Lambda_2)^\mu{}_\gamma (\Lambda_1)^\gamma{}_\nu$$

$$\Rightarrow (\Lambda_{21})^0{}_0 = (\Lambda_2)^0{}_\gamma (\Lambda_1)^\gamma{}_0 = (\Lambda_2)^0{}_0 (\Lambda_1)^0{}_0 + \sum_i (\Lambda_2)^0{}_i (\Lambda_1)^i{}_0 .$$

From the above

$$\left((\Lambda_1)^0{}_0\right)^2 \geq \sum_i \left((\Lambda_1)^i{}_0\right)^2, \ \left((\Lambda_2)^0{}_0\right)^2 \geq \sum_j \left((\Lambda_2)^0{}_j\right)^2 .$$

"Cauchy inequality":

$$\sum_{k=1}^n a_k^2 \sum_{\ell=1}^n b_\ell^2 \geq \left[\sum_{k=1}^n a_k b_k\right]^2 ,$$

$$\Rightarrow \left|(\Lambda_2)^0{}_0 (\Lambda_1)^0{}_0\right| \geq \sqrt{\sum_i \left((\Lambda_2)^0{}_i\right)^2 \sum_j \left((\Lambda_1)^j{}_0\right)^2} .$$

$$\underset{\text{Cauchy inequality}}{\geq} \sqrt{\left(\sum_i (\Lambda_2)^0{}_i (\Lambda_1)^i{}_0\right)^2} = \left|\sum_i (\Lambda_2)^0{}_i (\Lambda_1)^i{}_0\right| .$$

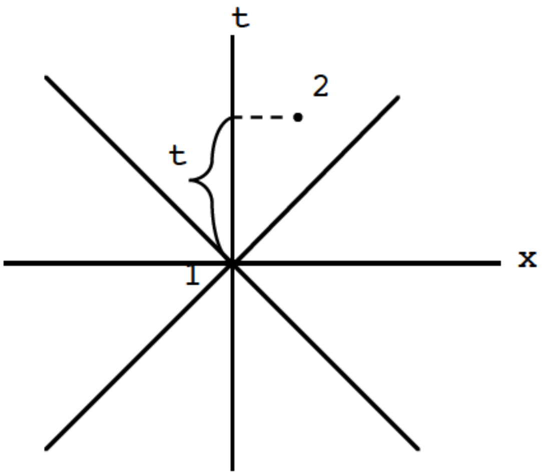
$\Rightarrow$ The sign of $(\Lambda_{21})^0{}_0$ is the same as $\left((\Lambda_2)^0{}_0 (\Lambda_1)^0{}_0\right)$.

Result:

No matter how many RLT's are performed, the result is still a RLT.

Another point:

Consider two timelike space-time events:



These events must have an invariant *time order* since they can have a cause and effect relationship. There can not be some observers who measure t > 0 and some who measure t < 0. That is, as a result of a RLT (boosts done on a material observer), we must show that the *time component* of a *timelike* 4-vector has an invariant *sign* (either + or -). (This will be done in a HW problem.) It is clear that *spacelike* events *can* have the sign of their time components changed by a RLT since there can be no causal relationship between them. As pointed out previously, no LT changes a timelike event into a spacelike one or vice versa.

There are 4 possible combinations of the discontinuous parameters det Λ and $\Lambda^0{}_0$. We classify these as

1. Proper orthochronous (RLT's): det $\Lambda = 1, \quad \Lambda^0{}_0 \geq 1.$

2. Improper orthochronous: det. $\Lambda = -1, \Lambda^0{}_0 \geq 1.$

3. Proper nonorthochronous: det $\Lambda = -1, \Lambda^0{}_0 \leq -1.$

4. Improper nonorthochronous: det. $\Lambda = -1, \Lambda^0{}_0 \leq -1.$

We have already talked about 1. What do the other transformations imply? Possibility 2 describes *space inversions* in addition to the usual RLT's. Example:

$$\Lambda_2 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{array}$$

Gives: $\vec{x} \rightarrow -\vec{x}, t \rightarrow t$. Any other Λ with $\det \Lambda = -1$, $\Lambda^0{}_0 \geq 1$ can then be written

$$\Lambda = \Lambda_{RLT}\,\Lambda_2\,. \tag{13.61}$$

Not new. Saw space inversions previously as part of 3-D orthogonal transformations. Takes us from one side to the other on the spacelike or timelike s^2 = const. surfaces in the lightcone diagram.

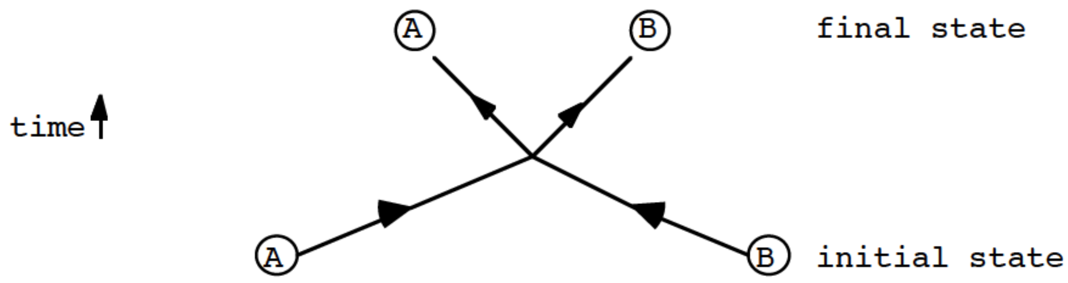
Possibility 4 describes something new: time reversal. Example:

$$\Lambda_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

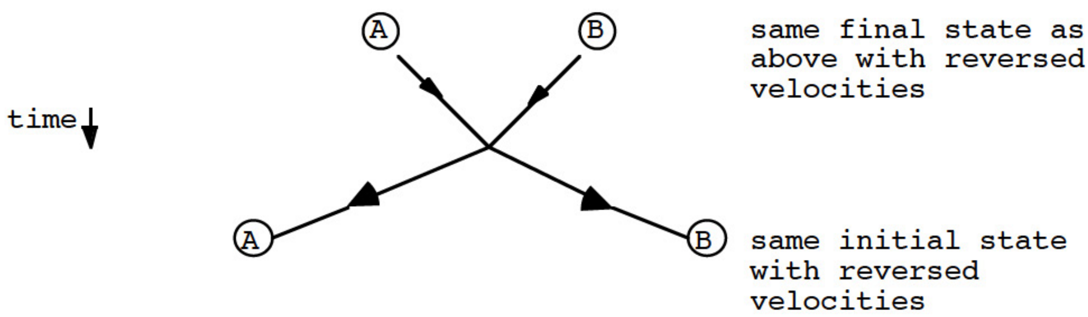
Gives $\vec{x} \to \vec{x}$, $t \to -t$. Any other Λ in this category can be written as

$$\Lambda = \Lambda_{RLT} \Lambda_4 . \tag{13.62}$$

Does to the time axis what possibility 2 did to space. Like space inversion, this is not a physical operation we can imagine carrying out on a material observer (*not* an RLT). Never-the-less, time and space inversions are valuable concepts, especially in quantum mechanics. To begin to understand why, it is helpful to consider the collision of particles (representing atoms or smaller objects). For example, consider the collision of two particles A and B:



If Newton's equations held at all levels of description, microscopically, as well as, macroscopically, it would be just as probable to see the time reserved reaction:



The reason for this is because Newton's equations of motion,

$$m_\alpha \frac{d^2\vec{x}_\alpha}{dt^2} = \vec{f}_\alpha, \qquad \left(\vec{f}_\alpha = \sum_{\beta<\alpha} U_{\alpha\beta}\left(\left|\vec{x}_\alpha - \vec{x}_\beta\right|\right).\right)$$

are invariant under $\vec{x}_\alpha \to \vec{x}_\alpha, t \to -t$. Modern microscopic theories of matter are also time-reversal invariant to a great extent; however, small violations of this symmetry have been seen experimentally. The fact that time-reversed reactions occur essentially as often as the original given one is an important fact of nature which is reflected in statistical mechanics and thermodynamics. However, on a macroscopic scale we know that many processes are not time-reversible; a glass shattering on the ground is one example. This is not a contradiction but an aspect of the behavior of large numbers of particles whose motions must be treated in a statistical sense. See the page from T.T. Lee's book "Particle Physics and Introduction of Field Theory" for a further illustration.

Possibility 3 represents *combined space and time reversals*. Example:

$$\Lambda_3 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{array}$$

Notice

$$\Lambda_3 = \Lambda_2\Lambda_4 .$$

This Λ gives $\vec{x} \to -\vec{x}, t \to -t$, and any other Λ with these characteristics can be written as usual,

$$\Lambda = \Lambda_{RLT}\,\Lambda_3 . \tag{13.63}$$

13.7 CONSEQUENCES OF RELATIVITY

Next, some consequences of special relativity. Consider again two space-time "events." Invariant associated with them is

$$\Delta s^2 = \sum_i \left(\Delta x_i\right)^2 - \left(c\Delta t\right)^2 .$$

Divide by $-c^2$:

$$\Delta\tau^2 \equiv -\frac{\Delta s^2}{c^2} = -\sum_i \left(\frac{\Delta x_i}{c}\right)^2 + \Delta t^2 . \tag{13.64}$$

$\Delta\tau$ (or just τ): "proper time". It is invariant and > 0 for timelike events. What is it's physical meaning? When two events happen at the same spatial position for some observer, $\Delta x^i = 0$ and

$$\Delta\tau = \Delta t .$$

Since the value $\Delta\tau$ is invariant, this means we can interpret proper time as follows:

Proper time: time interval measured by an observer who is at rest with respect to two events. (The two events take place at the same position.)

This means

$$\underset{\uparrow}{\Delta\tau} = \Delta t, \quad \text{``rest observer''}$$

same numerical value

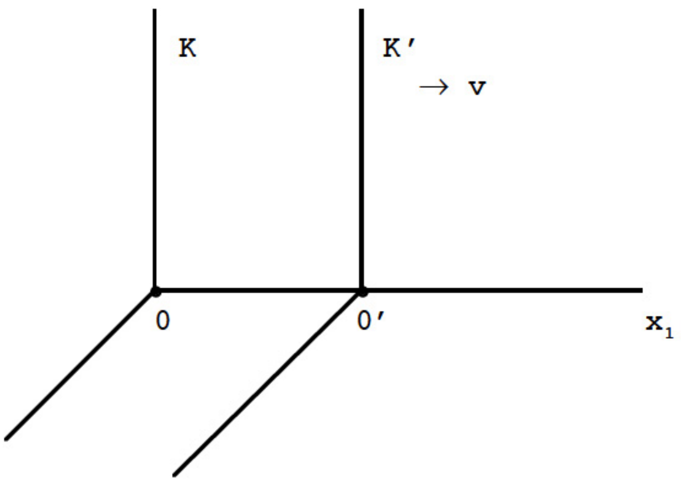
$$\Delta\tau' \equiv \sqrt{-\sum_i \left(\frac{\Delta x'^i}{c}\right)^2 + \Delta t'^2}\ , \text{ any other inertial observer}$$

Conclusion: Δt' > Δt.

The rest observer always measures the shortest time interval between these events. The name for this effect on time intervals is *time dilation*, because from the point of view of the moving observer, the time interval has increased or dilated. Can get a quantitative measure of this effect from our Lorentz transformation boost along x_1:

$$\begin{cases} x'_1 = \gamma\left(x_1 - vt\right), \\ x'_{2,3} = x_{2,3}\,, \\ t' = \gamma\left(t - x_1 \dfrac{v}{c^2}\right). \end{cases}$$

Reminder:



Situation:
First event at 0,0' when the axes coincide; then a new event at 0 every Δt seconds (as measured by K)

Event 1: $t' = t = 0$

Event 2: $t_1' = \gamma\left(t_1 - \overset{0}{\cancel{x}}_1 \frac{v}{c^2}\right) \qquad (x_1 = 0)$

Event 3: $t_2' = \gamma\, t_2$

•   •

•   •

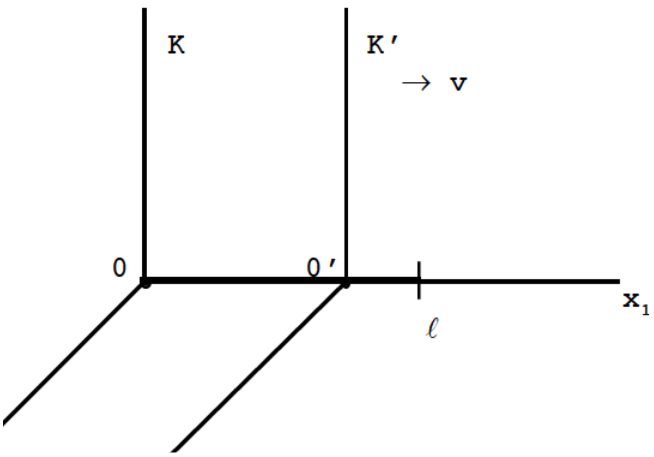
•   •

Relation between time intervals: Δt' = γΔt. We know that Δt = Δτ, so this gives

$$\Delta t' = \gamma\Delta\tau. \qquad \left(\gamma = \frac{1}{\sqrt{1-\beta^2}}\right) \tag{13.65}$$

Time dilation is well established experimentally.

Another consequence. Consider an object at rest in the K frame directed along x_1:



The observer in K measures the length of the object (which is at rest in his frame) and finds it to be ℓ. K', by moving along it's length, will measure it to be

$$\ell' = v\Delta t', \tag{13.66}$$

where Δt' is the time interval K' measures for passing the object. Now, the passing of 0' past the two ends of ℓ defines 2 space-time events, both of which take place at the same place, 0', according to K'. Thus, K' is measuring the proper time interval between these events,

$$\Delta t' = \Delta\tau. \tag{13.67}$$

Therefore from before, we know the time interval measured by K will be dilated:

$$\Delta t = \gamma \Delta\tau, \tag{13.68}$$

$$\Rightarrow \ell' = v\Delta\tau = \frac{v}{\gamma}\Delta t. \tag{13.69}$$

Of course, K has $\Delta t = \frac{\ell}{v}$, so

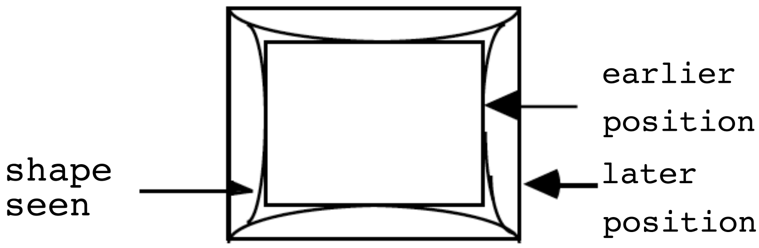
$$\ell' = \frac{v}{\gamma}\frac{\ell}{v} = \ell\sqrt{1 - \beta^2}. \tag{13.70}$$

$\Rightarrow \ell' < \ell$, "length contraction"

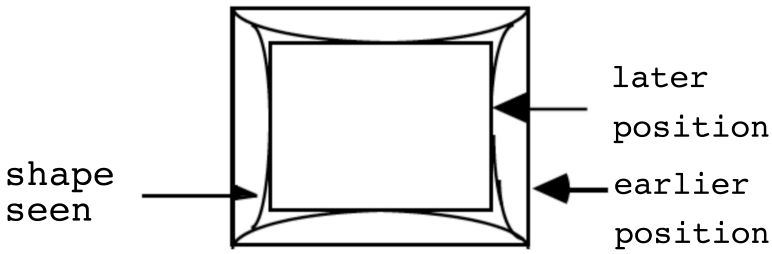
Proper length: length of an object which is at rest in an intertial observer's frame.

Interesting point: how an object moving with respect to an observer *looks like* is quite a different matter. Although the above considerations suggest the objects will appear contracted, in reality because of the finite speed of light, objects can *appear* to be rotated or have other distortions in appearance. For example, consider a cube moving directly away or toward an observer. It would actually look something like (consider light pulses which arrive at the same time):

Toward:



Away:



13.8 VELOCITY ADDITION LAW

Another consequence: velocity addition law is changed. After boost 1 by v_1:

$$\begin{cases} x_1' = \gamma_1 (x_1 - v_1 t), \\ x_{2,3}' = x_{2,3}, \\ t' = \gamma_1 \left(t - x_1 \frac{v_1}{c^2} \right). \end{cases} \tag{13.71}$$

Boost it again: (v_2)

$$\begin{cases} x_1'' = \gamma_2 (x_1' - v_2 t'), \\ x_{2,3}'' = x_{2,3}' = x_{2,3}, \\ t'' = \gamma_2 \left(t' - x_1' \frac{v_2}{c^2} \right). \end{cases} \tag{13.72}$$

In detail, this gives

$$x_1'' = \gamma_2\gamma_1\left[x_1\left(1 + \frac{v_1 v_2}{c^2}\right) - (v_1 + v_2)\,t\right], \tag{13.73}$$

$$x_1'' = \gamma_2\gamma_1\left[x_1\left(1 + \frac{v_1 v_2}{c^2}\right) - (v_1 + v_2)\,t\right], \tag{13.74}$$

Make it look like a *single* boost:

$$x_1'' = \gamma_1\gamma_2\left(1 + \frac{v_1 v_2}{c^2}\right)\left[x_1 - \left(\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\right)t\right], \tag{13.75}$$

$$t'' = \gamma_2\gamma_1\left[t\left(1 + \frac{v_1 v_2}{c^2}\right) - x_1\,\frac{(v_1 + v_2)}{c^2}\right]. \tag{13.76}$$

Exactly the same form as a single boost if

$$\gamma_{12} = \gamma_1\gamma_2\left(1 + \frac{v_1 v_2}{c^2}\right), \tag{13.77}$$

$$v_{12} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\ . \quad \leftarrow \text{new velocity addition law.} \tag{13.78}$$

Are they consistent?

$$\gamma_{12}^2 = \frac{1}{1 - \frac{v_{12}^2}{c^2}} = \frac{1}{1 - \frac{1}{c^2}\left(\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\right)^2} \quad \text{(just as we wanted)}$$

Example: $v_1 = .95c,\ v_2 = .95c$ (both in the same direction):

$$v_{12} = \frac{.95c + .95c}{1 + \frac{(.95c)^2}{c^2}} = .99868c\,.$$

$\Rightarrow$ Can not boost *through* v = c in a continuous fashion. This gives another perspective on the need to eliminate the earlier tachyonic F = -1 transformation, which required $\left|\frac{v}{c}\right| > 1$.

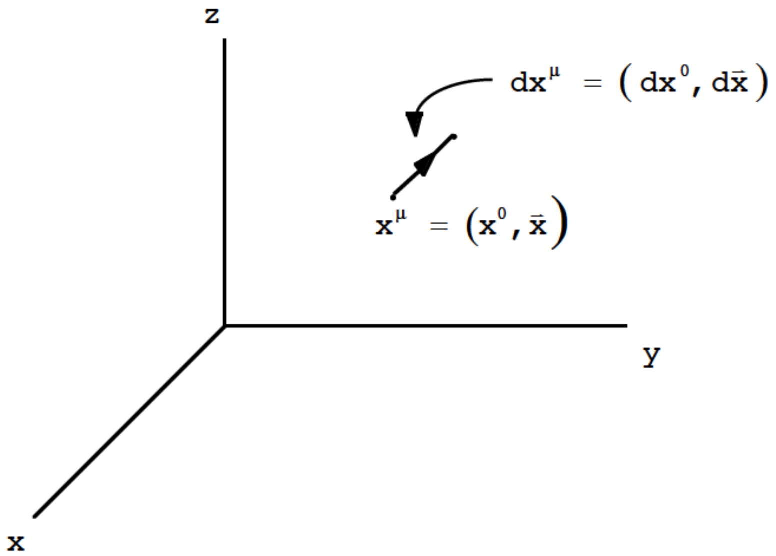
13.9 MOMENTUM AND ENERGY UNITED

Up to now, we have concentrated on the space-time aspects of relativity. However, relativity also has important implications for momentum and energy.

Use one of the above results:

$$\frac{dx^{\mu}}{d\tau} \text{ is a contravariant 4-vector}$$

Describe the motion of a particle in space and time:



By definition (remember $d\tau^2 = -\frac{ds^2}{c^2}$),

$$d\tau = \frac{1}{c}\sqrt{-(d\vec{x})^2 + (dx^0)^2}\,. \tag{13.79}$$

Must be a *timelike* vector if it describes the motion of a material observer. It is:

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{ds^2}{d\tau^2} = -c^2 < 0\,. \tag{13.80}$$

Write the components in a more ordinary notation:

$$\frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \frac{dt}{d\tau}\vec{v}\,, \tag{13.81}$$

$$\frac{d\tau}{dt} = \frac{1}{c}\sqrt{-\left(\frac{d\vec{x}}{dt}\right)^2 + c^2} = \sqrt{1-\beta^2}\,, \tag{13.82}$$

$$\Rightarrow \frac{d\vec{x}}{d\tau} = \gamma\vec{v} \underset{v<<c}{\longrightarrow} \vec{v}\,. \tag{13.83}$$

Time component:

$$\frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \gamma c \underset{v<<c}{\longrightarrow} c\,. \tag{13.84}$$

Re-verification of timelike character:

$$\left(\frac{d\bar{x}}{d\tau}\right)^2 = \frac{v^2}{1-\frac{v^2}{c^2}} \quad ,$$

$$\left(\frac{dx^0}{d\tau}\right)^2 = \frac{c^2}{1-\frac{v^2}{c^2}} \quad ,$$

$$\left(\frac{d\bar{x}}{d\tau}\right)^2 - \left(\frac{dx^0}{d\tau}\right)^2 = \frac{v^2}{1-\frac{v^2}{c^2}} - \frac{c^2}{1-\frac{v^2}{c^2}} = -c^2 \, .$$

Since the space component of $\frac{dx^\mu}{d\tau}$ reduces, in the v << c limit, to just $\vec{v}$, a natural definition of relativistic momentum is

$$p^\mu \equiv m\frac{dx^\mu}{d\tau} \, . \tag{13.85}$$

$$\Rightarrow \vec{p} = \gamma m\vec{v} \quad , \quad p^0 = \gamma mc \, .$$

Meaning of p^0? Notice that $p^0 > 0$ no matter which Lorentz frame we are in. (The sign of the time component of a timelike vector is unchanged under Lorentz transformations, which was a HW problem.) A particle's K.E. is also always > 0, so does $\frac{T}{c} = p^0$? Expand p^0 in a power series in $\frac{v}{c}$:

$$p^0 = \frac{1}{c}\left(mc^2 + \underbrace{\frac{1}{2}mv^2}_{\text{old K.E.}} + \ldots\right) . \tag{13.86}$$

Forces on us the realization that there is an energy associated not with *motion*, but with *mass*. Identify

$E^0 = mc^2$, “rest energy”

$E = p^0 c$. “total energy”

Einstein (“The Meaning of Relativity,” p.47): “Mass and energy are therefore essentially alike; they are only different expressions of the same thing.” Kinetic energy, T, is just the difference between total energy, E, and rest energy E_0:

$$\Rightarrow E = E_0 + T \quad , \quad T = E - E_0 = mc^2\,(\gamma - 1) \, . \tag{13.87}$$

Now

$$p^{\mu}\, p_{\mu} = -m^2 c^2,$$

so

$$-m^2 c^2 = \vec{p}^{\,2} - \frac{E^2}{c^2},$$

$$\Rightarrow E^2 = \vec{p}^{\,2} c^2 + m^2 c^4, \tag{13.88}$$

or, choosing the positive root,

$$E = +\sqrt{\vec{p}^{\,2} c^2 + m^2 c^4}\,, \qquad \left(\begin{array}{c} \text{old connection :} \\ E = \dfrac{\vec{p}^{\,2}}{2m}. \end{array}\right) \tag{13.89}$$

to be consitent with p^0 expressions. Now we realize we have done for $\vec{p}$ and E what has already been done for $\vec{x}$ and ct: unify them:

$$x^{\mu}x_{\mu} = \vec{x}^2 - x^{0^2} = \vec{x}^2 - c^2t^2 ,$$

$$p^{\mu}p_{\mu} = \vec{p}^2 - p^{0^2} = \vec{p}^2 - \frac{E^2}{c^2} .$$

The particle of light is called the photon; the connection between it's energy and momentum is given by (13.89) with m=0: $E = |\vec{p}|\,c$. It follows a "lightlike" trajectory ($d\tau = 0$ in (13.79)) and therefore it's inertial reference frame speed is always c.

13.10 FOUR SHORT POINTS

1. Transformation law for p^{μ}, p_{μ} same as for x^{μ}, x_{μ}:

$$p'^{\mu} = \Lambda^{\mu}{}_{\nu}\, p^{\nu} ,$$
$$p'_{\mu} = \Lambda_{\mu}{}^{\nu}\, p_{\nu} .$$

2. Conservation of both momentum and energy is now contained in the single statement:

$$\sum_{\substack{\text{incoming}\\ \text{particles}}} \left(p^{\mu}\right) = \sum_{\substack{\text{outgoing}\\ \text{particles}}} \left(p^{\mu}\right)$$

3. Because of equivalence of mass and energy, can express mass in units of $\frac{\text{energy}}{c^2}$:

$$\text{mass} = \frac{E}{c^2} = \underset{\substack{\uparrow\\ \text{common unit in particle}\\ \text{or nuclear physics}}}{\frac{\text{MeV}}{c^2}}$$

(1 MeV = 1.602×10^{-6} erg.)

[Often, one drops the $\frac{1}{c^2}$ and masses are often quoted simply in MeV.]

4. Equivalence between mass and energy is verified countless ways. For example, the "binding energy" of multiparticle systems appears as a deficit from the constituent particle masses. For the deuteron,

$$\text{deuteron mass: } 1875.7\ \frac{\text{MeV}}{c^2}$$

$$\left.\begin{array}{l}\text{proton : } 938.3\ \text{MeV}/c^2 \\ \text{neutron : } 939.6\ \text{MeV}/c^2\end{array}\right\}1877.9\ \frac{\text{MeV}}{c^2}$$

$$(1875.7 - 1877.9) = \underline{\underline{-2.2\ \text{MeV}/c^2}}\ \text{(the binding energy)}$$

13.11 PROBLEMS

1. Show that the 3-D wave equation (c=speed of light),

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f = 0,$$

(f is a scalar) is covariant under the Lorentz transformation ($\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{|v|}{c}$),

$$x' = \gamma(x - vt),$$

$$y' = y,$$

$$z' = z,$$

$$t' = \gamma(t - x\frac{v}{c^2}).$$

[Hint: Use the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'},$$

evaluating $\frac{\partial x'}{\partial x}$, $\frac{\partial t'}{\partial x}$, etc. from the transformation.]

2. Consider Lorentz transformations using index notation. Given

$$g_{\mu\nu}\,\Lambda^{\mu}{}_{\lambda}\,\Lambda^{\nu}{}_{\kappa} = g_{\lambda\kappa}\ ,\quad A'^{\sigma} = \Lambda^{\sigma}{}_{\tau}\,A^{\tau},\quad \Lambda_{\lambda}{}^{\gamma} = \frac{\partial x^{\gamma}}{\partial x'^{\lambda}},\quad \Lambda^{\lambda}{}_{\kappa} = \frac{\partial x'^{\lambda}}{\partial x^{\kappa}},$$

show:

(a) $\Lambda^{\nu}{}_{\lambda}\,\Lambda_{\mu}{}^{\lambda} = \delta^{\nu}_{\mu}$,

(b) $g_{\lambda\kappa}\,\Lambda_{\mu}{}^{\lambda}\,\Lambda_{\nu}{}^{\kappa} = g_{\mu\nu}$,

(c) $g^{\lambda\kappa}\,\Lambda^{\mu}{}_{\lambda}\,\Lambda^{\nu}{}_{\kappa} = g^{\mu\nu}$

3. Consider Lorentz transformations using matrix notation. Given that $\Lambda^T g\, \Lambda = g,\ g^T = g,\ gg = 1$ ("1" is the unit matrix), show that

(a) $(\Lambda^{-1})^T g\, \Lambda^{-1} = g,$

(b) $\Lambda\, g\, \Lambda^T = g.$

(c) Write (a) and (b) in component language.

4. For the tachyonic transformation,

$$x_1' = i\,x_1 \sin\psi - i x_4 \cos\psi,$$

$$x_4' = i x_1 \cos\psi + i\,x_4 \sin\psi.$$

$$\sin^2\psi = \frac{1}{1 - \frac{v^2}{c^2}}, \quad \left|\frac{v}{c}\right| > 1,$$

(a) Show that $x_1'^{\,2} + x_4'^{\,2} = -x_1^2 - x_4^2.$

(b) Demonstrate that under $\psi \to i\alpha$, where α is a real parameter, these become

$$x_1' = -x_1 \sinh\alpha - i x_4 \cosh\alpha,$$

$$x_4' = i x_1 \cosh\alpha - x_4 \sinh\alpha.$$

(c) Show $\Lambda^1{}_0 = \Lambda^0{}_1 \geq 1$. (Similar to $\Lambda^0{}_0 \geq 1$. for Lorentz boosts.)

(d) For the above, find $\det(\Lambda) = ?$.

5. Given the relativistic acceleration

$$\bar{\alpha} \equiv \frac{d\bar{u}}{d\tau},$$

where

$$\bar{u} \equiv \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\vec{v} = \frac{\vec{v}}{\sqrt{1-\beta^2}},$$

($\vec{v} = \dfrac{d\vec{x}}{dt}$ is the ordinary velocity; $\vec{\beta} \equiv \dfrac{\vec{v}}{c}$ show that

$$\vec{\alpha} = \frac{\dot{\vec{v}} + \frac{1}{c^2}\vec{v} \times (\vec{v} \times \dot{\vec{v}})}{(1-\beta^2)^2} \quad .$$

6. (a) Show that the relativistic form of Newton's second law is ($\vec{F} \equiv \dfrac{d\vec{p}}{dt}$)

$$\vec{F} = \frac{m\dot{\vec{v}}}{(1-\beta^2)^{3/2}} \quad ,$$

if $\vec{v}$ and $\dot{\vec{v}}$ are co-linear, and

b) $$\vec{F} = \frac{m\dot{\vec{v}}}{(1-\beta^2)^{1/2}} \quad ,$$

if $\vec{v}$ and $\dot{\vec{v}}$ are perpendicular. [Hint: Prob. 13.5 above.]

7. (a) A particle known as a muon is generated high in the Earth's atmosphere with a speed of 0.96c relative to the Earth. The muon's average lifetime, measured at rest, is 2.2×10^{-6} sec. How far does such a muon travel through the Earth's atmosphere before decaying? Briefly explain your calculation. ($c = 3 \times 10^8$ meters/sec.)

(b) A 30 yr. old physicist travels to a star, at rest with respect to the Earth, which is 10 light years distant. (1 light year = distance light travels in 1 year = 9.47×10^{15} meters.) His/Her rocket ship travels at 0.85 c relative to the Earth and star. How old will the physicist be on arriving at the star? Briefly explain again.

8. An astronaut brings an atomic clock on board the International Space Station (ISS), which is in orbit around the Earth at about 18,000 mi/hr. This clock is synchronized with an atomic clock on the Earth before the flight. If the clock is aloft in the ISS a month (take this as 30 days, measured let's say from the Earth's point of view), by how much time, in seconds, is the ISS clock advanced or delayed (which?) when compared to the Earth clock? (The speed of light is about 186,000 mi/sec.)

9. Show, by using conservation of energy and momentum, that a photon (a *massless* particle of light) with momentum $\vec{p}$ can not decay into two massive particles with momenta $\vec{p}_a$ and $\vec{p}_b$. (For simplicity, you may assume the masses of particles a and b are the same, although the result holds in general.) **Extra:** Can a photon decay into one massive and one massless particle?

10. Show that the time component of a timelike 4-vector t^{μ}, retains it's sign under a restricted Lorentz transformation. [Hint: Combine the timelike condition, $(ct^0)^2 > \sum (x^i)^2$, with a condition for a proper Lorentz transformation using the Cauchy inequality.]

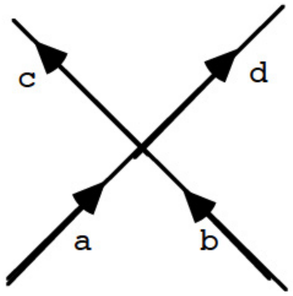
11. Define $u^{\mu} \equiv \dfrac{dx^{\mu}}{d\tau}$, $\alpha^{\mu} \equiv \dfrac{du^{\mu}}{d\tau}$. Show:

a) $\alpha^{\mu}\ u_{\mu} = 0$.
(i.e., the relativistic acceleration 4-vector is perpendicular to the relativistic velocity.)

b) α^{μ} is a *space like* 4-vector (assuming $\alpha^{\mu} \neq 0$).

[Way 1: Use (a) and construct a proof by contradiction by assuming α^{μ} is timelike or lightlike. Way 2: (Brute force) Construct α^{μ} explicitly and show that $\alpha^{\mu}\alpha_{\mu} > 0$.]

12. Given an elastic collision $a + b \rightarrow c + d$,



(Each line represents a particle with a mass m_a, m_b, m_c, or m_d)

show that the variables

$$s \equiv (p_a + p_b)(p_a + p_b)_\mu$$

$$t \equiv (p_c - p_a)(p_c - p_a)_\mu$$

$$u \equiv (p_d - p_a)(p_d - p_a)_\mu$$

satisfy the relation,

$$s + t + u = -c^2 \ (m_a{}^2 + m_b{}^2 + m_c{}^2 + m_d{}^2).$$

[These relatistic invariants are useful in describing the results of scattering events in different inertial frames.]

Other Problems

13. Show $\nabla_\mu A^\mu$ transforms as a scalar.

14. A particle of mass m is accelerated from rest with a constant force, F, in a straight line.

a) Show that if the final velocity is v $\left(\beta = \dfrac{|v|}{c}\right)$, then the distance traveled, D, is given by

$$D = \frac{mc^2}{F}\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right).$$

b) The acceleration takes place over a time period, T. Show that this time period is given by

$$cT = \frac{mc^2}{F}\sqrt{(1 + \frac{FD}{mc^2})^2 - 1}\,.$$

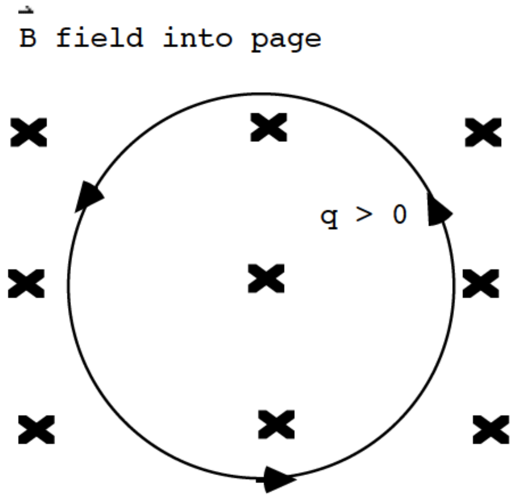
[Hint: integrate the result of part (a).]

15. Consider a relativistic particle of mass m and charge q in a circular orbit of radius R. The force on the charge is central and is given by

$$\vec{F} = -\frac{q^2}{R^2}\hat{r},$$

where the unit vector $\hat{r}$ points from the force center to the charge. Find the relativistic formula giving $\beta = \frac{|v|}{c}$ of the particle as a function of m, q, and R.

16. A relativistic particle of mass m and charge q is moving with velocity v in a plane perpendicular to a uniform magnetic field, $\vec{B}$.



Using the Lorentz force law, find the radius of the orbit in terms of v, q, B and m. In particular, show that the radius for ultrarelativistic motion ($E >> mc^2$) is $R = \frac{E}{|q\vec{B}|}$, where E is the total energy (rest mass plus kinetic) of the particle.

17. A coordinate system K' moves with velocity $\vec{v}$ relative to another system, K. In K' a particle has a velocity $\vec{u}'$ and an acceleration $\vec{a}'$, whereas in K these quantities are $\vec{u}$ and $\vec{a}$. Show that in K the observed component of acceleration of the particle along $\vec{v}$, $\vec{a}_{||}$, is given by ($\beta = \frac{|v|}{c}$),

$$\vec{a}_{||} = = \frac{(1-\beta^2)^{3/2}}{(1+\frac{\vec{v}\cdot\vec{u}'}{c^2})^3}\vec{a}'_{||}.$$

[Hint: First get the relation between $\vec{u}_{||}$ and $\vec{u}'_{||}$.]

18. There are actually three classes of transformations consistent with the invariance of the statement (μ = 1,2,3,4 notation; $x'_4 = ict'$),

$$\sum_{\mu} \bar{x}_{\mu}^{\,2} = 0\,, \tag{1}$$

for the coordinates, $\bar{x}_{\mu}$, of a lightfront. These are:

$$\text{I.} \quad x'_{\mu} = \sum_{\mu} \lambda_{\mu\nu} x_{\nu}\,, \text{ where } \sum_{\mu} \lambda_{\mu\nu}\lambda_{\mu\gamma} = \delta_{\nu\gamma}\,,$$

$$\text{II.} \quad x'_{\mu} = \lambda x_{\mu}\,, \text{ ("dilations")}$$

$$\text{III.} \quad x'_{\mu} = \frac{x_{\mu} + x^2 c_{\mu}}{(1 + 2(c \cdot x) + x^2 c^2)}\,, \; c_{\mu} \text{ arbitrary. } \left((c \cdot x) \equiv \sum_{\mu} c_{\mu} x_{\mu}\,,\right.$$
$$\left. x^2 \equiv \sum_{\mu} x_{\mu} x_{\mu}\,, \text{ and where } (1 + 2(c \cdot x) + x^2 c^2) \neq 0\right) \text{ ("conformal transformations")}$$

Show that cases II and III are consistent with the invariance of equation (1). (Case I already considered in the text.) Why are Cases II and III not considered further in the text for relativity?

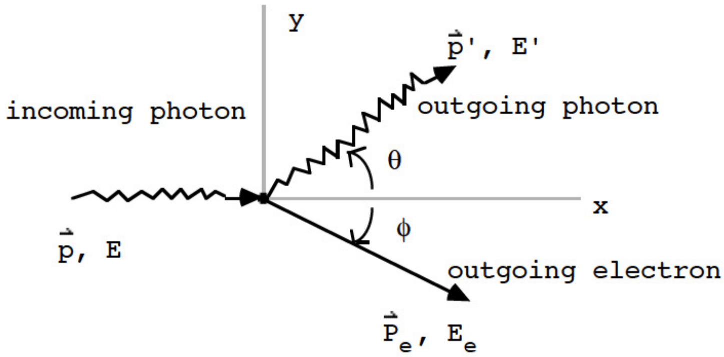
19. In prob. 13.9, we saw that a photon can not decay into 2 massive particles (or into 1 massless and one massive particle). Now examine two other cases:

(i) $m_A \rightarrow \gamma + \gamma$ (particle of mass mA decays into two photons)

(ii) $m_A \rightarrow m_B + \gamma$ (particle of mass mA decays into particle of mass $m_B < m_A$ and a single photon)

Do energy and momentum conservation allow these processes to occur? If not, show it. If allowed, solve for the magnitude of photon momentum, $p = E/c$, in terms of m_A in (i) and in terms of mA and m_B in case (ii). (Assume particle m_A is at rest in both cases.)

20. Consider so-called Compton scattering on an initially stationary electron of mass m (use $p \equiv |\vec{p}|$, $p' \equiv |\vec{p}'|$, $P_e \equiv |\vec{P}_e|$):



(a) Given the energy, E, and momentum, $\vec{p}$, of the incoming photon along the x-direction, write all the equations which follow from momentum and energy conservation.

(b) From these equations show that

$$-mc\left(-\frac{1}{p}+\frac{1}{p'}\right) = 1 - \cos\theta \,.$$

This determines the momentum, p', of the scattered photon in terms of the initial momentum, p, and the scattering angle, θ. [Hint: Get two expressions for the square of the electron momentum, $P_e^{\,2}$, and set them equal.]

ENDNOTES

1. Notice that we could replace condition 3 with the alternate condition that the system of equations, (12.17), for small oscillations yield only real nonzero eigenfrequencies, ω^r. This was the same condition for stability seen in Ch.2.
2. Note: We are using "x" here to denote coordinates, but of course in the previous treatment $\mu = 1,2,3,4$ and $x_4 = ict$, whereas here $\mu = 0,1,2,3$ and $x_0 = ct$ ($x_0 = -ct$).